On chromatic number of some Kneser hypergraphs

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Joint work with Meysam Alishahi

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Introduction

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Died	16 February 2004 (aged 76)

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Definition of the Kneser hypergraph $\mathrm{KG}^r(n,k)$

An r-uniform hypergraph with the vertex set $\binom{[n]}{k}$ and the edge set $\{\{e_1,\ldots,e_r\} : \ |e_i| = k, \ e_i \subseteq [n]$ and $e_i \cap e_j = \varnothing, \forall i \neq j \in [r]\}$.

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Theorem A (Lovász (1978) (for $r = 2$) and Alon, Frankl, and Lovász (1986))

$$\chi(\mathrm{KG}^r(n,k)) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil.$$

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$$cd^r(\mathcal{H}) = \min\{|Y| : Y \subseteq V(\mathcal{H}), \chi(\mathcal{H}[V(\mathcal{H}) \setminus Y]) \leq r\}.$$

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The sequence $x_{i_1}, \ldots, x_{i_m}$ is called *alternating* whenever $1 \leq i_1 < \cdots < i_m \leq n$, $x_{i_j} \neq 0, \forall j \in [m]$ and $x_{i_j} \neq x_{i_{j+1}}, \forall j \in [m-1]$.
Let $\mathbb{Z}_r = \{\omega, \omega^2, \ldots, \omega^r\}$ be a multiplicative cyclic group of order r with generator ω. For each $X \in (\mathbb{Z}_r \cup \{0\})^n$, $alt^r(X)$ is the length of longest alternating subsequence of X. Also, $alt(\mathbf{0}) = 0$.

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Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph, and $\sigma : [n] \longrightarrow V(\mathcal{H})$ be a bijection.

Definition of r-alternation number of $\mathcal{H}$ with respect to the bijection σ

$$alt_\sigma^r(\mathcal{H}) = \max\left\{alt(X) \colon X \in (\mathbb{Z}_r \cup \{0\})^n \text{ and } E(\mathcal{H}[\sigma(X^i)]) = \varnothing \text{ for each } i \in [r]\right\}.$$

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Define $alt^r(\mathcal{H}) = \min_{\sigma} alt^r_\sigma(\mathcal{H}), \quad \forall \sigma : [n] \longrightarrow V(\mathcal{H})$.

Theorem C (Alishahi and Hajiabolhassan (2015))

$$\chi(\mathrm{KG}^r(\mathcal{H})) \geq \left\lceil \frac{|V(\mathcal{H})| - alt^r(\mathcal{H})}{r-1} \right\rceil .$$

Definition of an equitable r-coloring

A propoer r-coloring such that the sizes of color classes differ by at most one.

Defenition of r-equitable colorability defect

$$ecd^r(\mathcal{H}) = n - \max\left\{\sum_{j=1}^{r}|N_j| \;:\; N_j \subseteq [n], \quad \begin{array}{c} N_1,\ldots,N_r \\ \text{are equitable pairwise disjoint sets and} \\ E(\mathcal{H}[N_j]) = \varnothing, \forall j \in [r]. \end{array}\right\}$$

Some results on the Kneser hypergraph coloring

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Theorem 1 (A. and Alishahi (2018))

For any hypergraph $\mathcal{H}$ and any integer r with $r \geq 2$, we have

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Proof of Theorem 1

Lemma 1

For any hypergraph $\mathcal{H} = ([n], E)$ and prime number p,

$$\chi(\mathrm{KG}^p(\mathcal{H})) \geq \left\lceil \frac{\mathrm{ecd}^p(\mathcal{H})}{p-1} \right\rceil.$$

Lemma 2

Let r, r', r'' be positive integers, where $r', r'' \geq 2$ and $r = r'r''$. If Theorem 1 holds for r' and r'', then it holds for r.

Let $\mathcal{H} = ([n], E)$ be a hypergraph, n be a positive integer and p be a prime number.

Let $c : E(\mathcal{H}) \longrightarrow [t]$ be a proper coloring of $\mathrm{KG}^p(\mathcal{H})$.

Sketch the proof of Lemma 1

Let $\mathcal{H} = ([n], E)$ be a hypergraph, n be a positive integer and p be a prime number.

Let $c : E(\mathcal{H}) \longrightarrow [t]$ be a proper coloring of $\mathrm{KG}^p(\mathcal{H})$.

Define a $\mathbb{Z}_p$-simplicial map $\gamma : \mathrm{sd}(\sigma_0^{p-1})^{*n} \longrightarrow \mathbb{Z}_p \times [m]$ utilizing the function $l(-)$ introduced by Alishahi and some sign functions proposed by Meunier.

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- Apply Dold's theorem which asserts that if there is a $\mathbb{Z}_p$-simplicial map from a free $\mathbb{Z}_p$ space C to a free $\mathbb{Z}_p$ space K, then dimension of K is larger than connectivity of C.

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Proof of Lemma 2

For a hypergraph $\mathcal{H} = ([n], E)$ and positive integers r and C, define $\mathcal{T}_{\mathcal{H},C,r}$ to be the hypergraph with the vertex set $V(\mathcal{H})$ and the edge set $E(\mathcal{T}_{\mathcal{H},C,r}) = \left\{ V \subseteq V(\mathcal{H}) : \mathrm{ecd}^r(\mathcal{H}[V]) > (r-1)C \right\}$.

Lemma 3

Let r', r'' be two positive integers For any hypergraph $\mathcal{H} = ([n], E)$, the following inequality holds

$$\mathrm{ecd}^{r'r''}(\mathcal{H}) \leq r''(r'-1)C + \mathrm{ecd}^{r''}(\mathcal{T}_{\mathcal{H},C,r'}).$$

Proof of lemma 2.

On the contrary, suppose that there exists a proper coloring $c : E(\mathcal{H}) \to [C]$ of $\mathrm{KG}^r(\mathcal{H})$ for which $\mathrm{ecd}^r(\mathcal{H}) > (r-1)C$. Lemma 3 leads us to the inequality $(r''-1)C < \mathrm{ecd}^{r''}(\mathcal{T}_{\mathcal{H},C,r'})$, which implies that $\chi(\mathrm{KG}^{r''}(\mathcal{T}_{\mathcal{H},C,r'})) > C$, contradicting the fact that c is a proper coloring for $\mathrm{KG}^r(\mathcal{H})$.

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Let n,k and a be positive integers, where $n>a$.

Definition of hypergraph $\mathcal{H}(n,k,a)$

A hypergraph with vertex set $[n]$ and whose edge set is defined as follows:

$$E(\mathcal{H}(n,k,a))=\{e\subseteq[n]\colon |e|=k \text{ and } e\not\subseteq\{n-a+1,\ldots,n\}\}.$$

Lemma 4

Let n,k,r, and a be positive integers with $k,r\geq 2$, $n>a,rk$. Then, we have

$$\chi(\mathrm{KG}^r(n,k,a))\leq\left\lceil\frac{n-\max\{rk-1,a+r-1\}}{r-1}\right\rceil+1.$$

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Lemma 5

Let $n, k, r,$ and a be positive integers a with $k, r \geq 2$, $n \geq \max\{a+1, rk\}$. Then

$$\mathrm{ecd}^r(\mathcal{H}(n,k,a)) = \begin{cases} n - r(k-1) & a \leq k-1 \\ n - r(k-1) - \left\lfloor \frac{a}{k} \right\rfloor & k \leq a \leq rk-2 \\ n - a & a \geq rk-1, \end{cases}$$

$$\mathrm{cd}^r(\mathcal{H}(n,k,a)) = \begin{cases} n - r(k-1) & a \leq k-1 \\ \max\{0, n-(r-1)(k-1)-a\} & a \geq k \end{cases}$$

and

$$n - alt^r(\mathcal{H}(n,k,a)) \leq \begin{cases} n - \max\{2(k-1),\ a\} & r = 2 \\ n - \max\{3(k-1),\ a+1\} & r = 3 \\ n - \max\{r(k-1),\ a+b\} & r \geq 4, \end{cases}$$

where $b = \min\{n-a, (r-2)(k-1)\}$.

Proposition 1 (A. and Alishahi (2018))

Let n, r, k, and a be positive integers, where $r, k \geq 2$, $n > a$, and $n \geq rk$. Then we have

$$\chi(\mathrm{KG}^r(n,k,a)) = \left\lceil \frac{n - \max\{r(k-1), a\}}{r-1} \right\rceil,$$

provided that either $a \leq 2(k-1)$ or $a \geq rk-1$. Moreover, for $2k-1 \leq a \leq rk-2$, we have

$$\left\lceil \frac{n - r(k-1) - \lfloor \frac{a}{k} \rfloor}{r-1} \right\rceil \leq \chi(\mathrm{KG}^r(n,k,a)) \leq \left\lceil \frac{n - \max\{r(k-1), a\}}{r-1} \right\rceil.$$

Observation 1

Let $\mathcal{H} = \mathcal{H}(n,k,a)$. For $r \geq 4$, $a \geq rk-1$, and $n \geq (r-1)(k-1)+a$, the values of $\frac{1}{r-1}\left(\mathrm{ecd}^r(\mathcal{H}) - \mathrm{cd}^r(\mathcal{H})\right)$ and $\frac{1}{r-1}\left(\mathrm{ecd}^r(\mathcal{H}) - (n - alt^r(\mathcal{H}))\right)$ can be made as large as desired by making k large enough.

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Cojecture

Let n, r, k, and a be positive integers, where $r, k \geq 2$, $n > a$, and $n \geq rk$. We have

$$\chi(\mathrm{KG}^r(n, k, a)) = \left\lceil \frac{n - \max\{r(k-1), a\}}{r-1} \right\rceil.$$

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THANKS FOR YOUR ATTENTION.