Augmented Lagrangian-type Preconditioning

for an HDG Discretization of the Oseen Equations

Abdullah Ali Sivas, Sander Rhebergen

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Table of contents

The Oseen Equations

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\begin{aligned}
-\nu\nabla^2\vec{u} + (\vec{w}\cdot\nabla)\vec{u} + \nabla p &= \vec{f} && \text{in } \Omega,\\
\nabla\cdot\vec{u} &= 0 && \text{in } \Omega,\\
\vec{u} &= \vec{g}_D && \text{on } \partial\Omega_D,\\
\nu\frac{\partial\vec{u}}{\partial\vec{n}} - \vec{n}p &= 0 && \text{on } \partial\Omega_N,
\end{aligned}
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$\vec{w}$: the given solenoidal field ($\nabla\cdot\vec{w} = 0$)

$\vec{u}$: the velocity of a fluid

p: the pressure

Reynolds number $Re := \frac{WL}{\nu}$

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Why are these equations interesting?

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$$\begin{aligned} -\nu\nabla^2\vec{u}^n + (\underbrace{\vec{u}^{n-1}}_{=\vec{w}}\cdot\nabla)\vec{u}^n + \nabla p^n &= \vec{f} && \text{in } \Omega, \\ \nabla\cdot\vec{u}^n &= 0 && \text{in } \Omega, \\ \vec{u}^n &= \vec{g}_D && \text{on } \partial\Omega_D, \\ \nu\frac{\partial\vec{u}^n}{\partial\vec{n}} - \vec{n}p^n &= 0 && \text{on } \partial\Omega_N, \end{aligned}$$

Hybridizable Discontinuous Galerkin Methods

Idea of Hybridization

Consider the *somewhat* general problem

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } \Omega \\ u &= g_D \quad \text{on } \partial\Omega_D. \end{aligned}$$

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$$\mathcal{L}u = f \quad \text{in } \Omega$$
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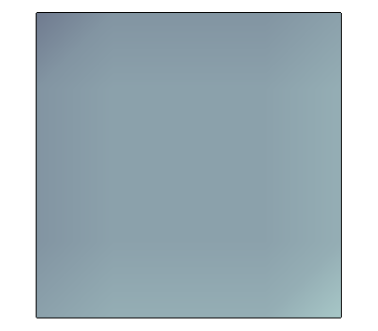
For simplicity, let $\Omega = [0,1]^2$.



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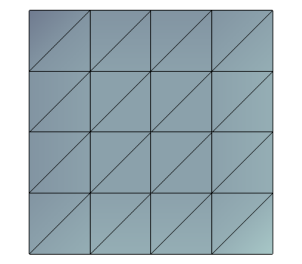
Separate the domain $\mathcal{K}$ from its skeleton $\mathcal{E}$.



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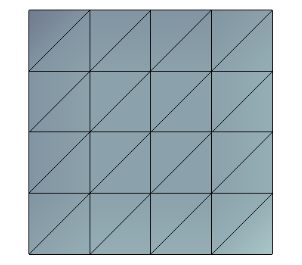

Rewrite the problem for each $K \in \mathcal{K}$

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } K \\ u &= \hat{u} \quad \text{on } \partial K \subset \mathcal{E}, \end{aligned}$$

with the transmission conditions $[[\hat{u}]] = 0$ over interior faces and $\hat{u} = g_D$ on boundary.

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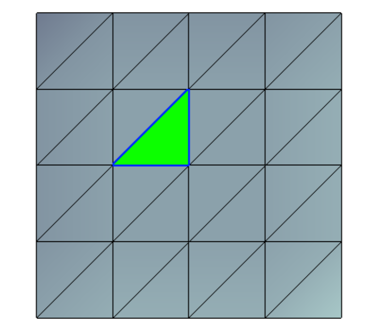
K: green triangle, ∂K blue outline



Idea of Hybridization

The problem

$$\begin{aligned} \mathcal{L}u = f &\quad \text{in } K \\ u = \hat{u} &\quad \text{on } \partial K \subset \mathcal{E}. \end{aligned}$$

complete with the transmission conditions is called *the local problem*.

Why Hybridize Different FEMs?

Continuous Galerkin:

[1] Rhebergen and Wells, "A Hybridizable Discontinuous Galerkin Method for the Navier–Stokes Equations with Pointwise Divergence-Free Velocity Field"

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Continuous Galerkin:

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  - pressure-robustness

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Solution of the Linear System

Matrix Form

$$\begin{bmatrix} N_{uu} & N_{u\bar{u}} & B_{up} & B_{u\bar{p}} \\ N_{\bar{u}u} & N_{\bar{u}\bar{u}} & 0 & B_{\bar{u}\bar{p}} \\ C_{pu} & 0 & 0 & 0 \\ C_{\bar{p}u} & C_{\bar{p}\bar{u}} & 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ \bar{U} \\ P \\ \bar{P} \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix Form

Dirichlet boundary conditions are enforced strongly,

$$\begin{bmatrix} N_{uu} & \widetilde{N_{u\bar{u}}} & B_{up} & B_{u\bar{p}} \\ \widetilde{N_{\bar{u}u}} & \widetilde{N_{\bar{u}\bar{u}}} & 0 & \widetilde{B_{\bar{u}\bar{p}}} \\ C_{pu} & 0 & 0 & 0 \\ C_{\bar{p}u} & \widetilde{C_{\bar{p}\bar{u}}} & 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ \bar{U} \\ P \\ \bar{P} \end{bmatrix} = \begin{bmatrix} \widetilde{F} \\ G \\ 0 \\ \Phi \end{bmatrix}$$

Matrix Form

Due to hybridization, N_{uu}, B_{up} and C_{pu} are block-diagonal matrices. Eliminate U and P to get

$$\begin{bmatrix} S_{\bar{u}\bar{u}} & S_{\bar{u}\bar{p}} \\ S_{\bar{p}\bar{u}} & S_{\bar{p}\bar{p}} \end{bmatrix} \begin{bmatrix} \bar{U} \\ \bar{P} \end{bmatrix} = \begin{bmatrix} L_{\bar{u}} \\ L_{\bar{p}} \end{bmatrix},$$

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with

$$S_{\bar{u}\bar{u}} = N_{\bar{u}\bar{u}} - N_{\bar{u}u} N_{uu}^{-1} N_{u\bar{u}} + N_{\bar{u}u} N_{uu}^{-1} B_{up} \left(C_{pu} N_{uu}^{-1} B_{up} \right)^{-1} C_{pu} N_{uu}^{-1} N_{u\bar{u}},$$
$$S_{\bar{u}\bar{p}} = B_{\bar{u}\bar{p}} - N_{\bar{u}u} N_{uu}^{-1} B_{u\bar{p}} + N_{\bar{u}u} N_{uu}^{-1} B_{up} \left(C_{pu} N_{uu}^{-1} B_{up} \right)^{-1} C_{pu} N_{uu}^{-1} B_{u\bar{p}},$$
$$S_{\bar{p}\bar{u}} = C_{\bar{p}\bar{u}} - C_{\bar{p}u} N_{uu}^{-1} N_{u\bar{u}} + C_{\bar{p}u} N_{uu}^{-1} B_{up} \left(C_{pu} N_{uu}^{-1} B_{up} \right)^{-1} C_{pu} N_{uu}^{-1} N_{u\bar{u}},$$
$$S_{\bar{p}\bar{p}} = -C_{\bar{p}u} N_{uu}^{-1} B_{u\bar{p}} + C_{\bar{p}u} N_{uu}^{-1} B_{up} \left(C_{pu} N_{uu}^{-1} B_{up} \right)^{-1} C_{pu} N_{uu}^{-1} B_{u\bar{p}},$$
$$L_{\bar{u}} = G - N_{\bar{u}u} N_{uu}^{-1} F + N_{\bar{u}u} N_{uu}^{-1} B_{up} \left(C_{pu} N_{uu}^{-1} B_{up} \right)^{-1} C_{pu} N_{uu}^{-1} F \text{ and}$$
$$L_{\bar{p}} = \Phi - C_{\bar{p}u} N_{uu}^{-1} F + C_{\bar{p}u} N_{uu}^{-1} B_{up} \left(C_{pu} N_{uu}^{-1} B_{up} \right)^{-1} C_{pu} N_{uu}^{-1} F.$$

Matrix Form

Other options

- Only eliminate U, simpler form and very useful for the Stokes problem[2]
- Do not eliminate at all, large system size but easier to manipulate

[2]Rhebergen and Wells, "Analysis of a hybridized/interface stabilized finite element method for the Stokes equations"

Preliminaries

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R optimal

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$\kappa(A)$ depends on h and Re

R optimal , $\kappa(R^{-1}A)$ parameter independent.

Perfect Preconditioner

Let's write the linear system in more compact form

$$\begin{bmatrix} N & B \\ C & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F \\ L \end{bmatrix}.$$

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Turns out the preconditioner

$$\begin{bmatrix} \mathrm{N} & \mathrm{B} \\ 0 & \mathrm{C}\mathrm{N}^{-1}\mathrm{B} \end{bmatrix}$$

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Turns out the preconditioner

$$\begin{bmatrix} N & B \\ 0 & S \end{bmatrix}$$

is the "perfect" preconditioner and iterative solvers (e.g. GMRES) converges in two iterations[3].

[3] Elman, Silvester and Wathen, "Finite elements and fast iterative solvers: with applications in incompressible fluid dynamics"

Some Approaches

- Approximate the Schur complement in some way
  - SIMPLE
  - Pressure Convection-Diffusion, Least Squares Commutator
  - ...
- Modify the system while keeping the problem same, e.g. Augmented Lagrangian

Augmented Lagrangian - Continuous

Consider again

$$\begin{aligned} -\nu\nabla^2\vec{u} + (\vec{w}\cdot\nabla)\vec{u} + \nabla p &= \vec{f} && \text{in } \Omega, \\ \nabla\cdot\vec{u} &= 0 && \text{in } \Omega, \\ \vec{u} &= \vec{g}_D && \text{on } \partial\Omega_D, \\ \nu\frac{\partial\vec{u}}{\partial\vec{n}} - \vec{n}p &= 0 && \text{on } \partial\Omega_N, \end{aligned}$$

Augmented Lagrangian - Continuous

But add the grad-div term

$$\begin{aligned} \gamma\nabla(\nabla\cdot\vec{u}) - \nu\nabla^2\vec{u} + (\vec{w}\cdot\nabla)\vec{u} + \nabla p = \vec{f} \qquad & \text{in } \Omega, \\ \nabla\cdot\vec{u} = 0 \qquad & \text{in } \Omega, \\ \vec{u} = \vec{g}_D \qquad & \text{on } \partial\Omega_D, \\ \nu\frac{\partial\vec{u}}{\partial\vec{n}} - \vec{n}p = 0 \qquad & \text{on } \partial\Omega_N, \end{aligned}$$

with penalty coefficient $\gamma \geq 0$.

Notice that $\nabla\cdot\vec{u} = 0 \implies \gamma\nabla(\nabla\cdot\vec{u}) = 0 \ \forall\gamma$.

Augmented Lagrangian-like Preconditioner

In the matrix form

$$\begin{bmatrix} N & B \\ C & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F \\ L \end{bmatrix}$$

Augmented Lagrangian-like Preconditioner

Add

$$\begin{bmatrix} \gamma \mathrm{B}\mathrm{W}^{-1}\mathrm{C} + \mathrm{N} & \mathrm{B} \\ \mathrm{C} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{U} \\ \mathrm{P} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathrm{F}} \\ \mathrm{L} \end{bmatrix}$$

for some invertible matrix W.

Augmented Lagrangian-like Preconditioner

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for some invertible matrix W.

Then the "perfect" preconditioner is

$$\begin{bmatrix} \mathrm{N} & \mathrm{B} \\ 0 & \mathrm{C}(\gamma \mathrm{B}\mathrm{W}^{-1}\mathrm{C} + \mathrm{N})^{-1}\mathrm{B} \end{bmatrix}$$

Augmented Lagrangian-like Preconditioner

Add

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Then the "perfect" preconditioner is

$$\begin{bmatrix} \mathrm{N} & \mathrm{B} \\ 0 & \mathrm{S}_\gamma \end{bmatrix}$$

Inverse of S_γ

Lemma Assuming $\mathrm{C}(\gamma \mathrm{B} \mathrm{W}^{-1}\mathrm{C} + \mathrm{N})^{-1}\mathrm{B}$, $\gamma \mathrm{B} \mathrm{W}^{-1}\mathrm{C} + \mathrm{N}$, N and $\mathrm{C}\mathrm{N}^{-1}\mathrm{B}$ are invertible, then the inverse of S_γ is given by

$$\mathrm{S}_\gamma^{-1} = \mathrm{S}^{-1} + \gamma \mathrm{W}^{-1}$$

where $\mathrm{S} = \mathrm{C}\mathrm{N}^{-1}\mathrm{B}$.

Proof Benzi and Olshanskii, "An Augmented Lagrangian Based Approach to the Oseen Problem"

Advantages and Disadvantages

✓ Take $\gamma \gg 1$, $\mathrm{S}_\gamma^{-1} \approx \gamma \mathrm{W}^{-1}$ and pick $\mathrm{W} = \mathrm{M}_P$, which is block diagonal and hence, easily solved.

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- × $\gamma \mathrm{B} \mathrm{W}^{-1} \mathrm{C} + \mathrm{N}$ is now closer to grad-div problems, which have large null spaces

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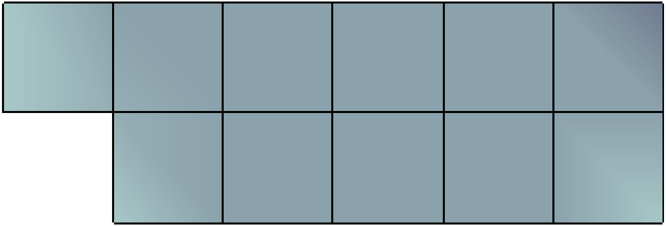
- × $\gamma \mathrm{B} \mathrm{W}^{-1} \mathrm{C} + \mathrm{N}$ is now closer to grad-div problems, which have large null spaces
- × Have to find optimal γ for each problem separately

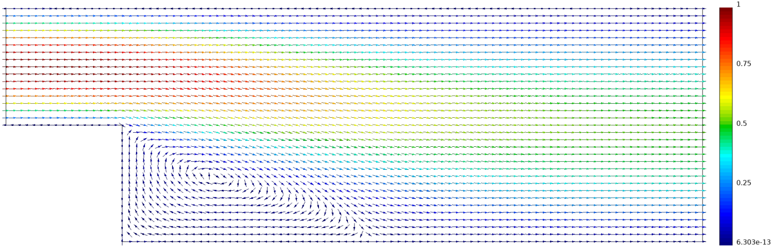
Numerical Results

The Backward Facing Step Problem

The Backward Facing Step Problem



The Backward Facing Step Problem



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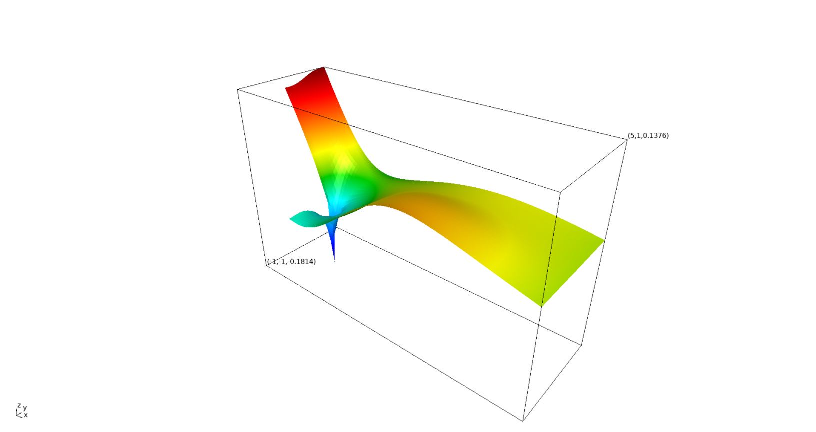

The Methodology

Borrowing notation from MATLAB,

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For Augmented Lagrangian preconditioning, modify `A` by adding $\gamma \mathrm{B} \mathrm{W}^{-1} \mathrm{C}$ for various γ.

Results

Table 1: Convergence of GMRES

		AL				ILU	PARASAILS
	γ	1	10	100	1000	-	-
ν	*l*	#its	#its	#its	#its	#its	#its
	2	34	18	9	6	20	51
1/5	3	31	15	9	6	67	171
	4	27	14	8	6	196	927
	2	61	26	13	7	13	60
1/50	3	49	25	13	7	28	215
	4	47	23	12	7	89	688

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- Some side effects like stronger enforcement of mass conservation

Thanks for listening!