Surrounding Tomaszewskis Conjecture

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Conjecture...

In 1986, Boguslav Tomaszewski posed the following problem:

- Consider a real vector $\alpha \in \mathbb{R}^n$ such that $\sum_{i=1}^n \alpha_i^2 = 1$.
- Of the 2^n expressions

$$S = |\pm \alpha_1 \pm \cdots \pm \alpha_n|,$$

can there be more > 1 than ≤ 1?

Conjecture

For every vector α, at least half of the expressions are ≤ 1.

Main formulation

The most common formulation of the problem is using the language of probability.

- Consider a random vector $x \in \{-1, 1\}^n$. Then,

$$P[|\alpha \cdot x| \leq 1] \geq \frac{1}{2}$$

.

- The equality stands for $\alpha = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, \dots\right)$, or just $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Past Work

- First result was acquired in 1992 by Holzman and Kleitman. They proved that:

$$P[|\alpha \cdot x| < 1] \geq \frac{3}{8}$$

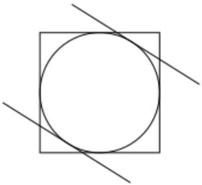
  The equality stands for $\alpha = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.
- The result is sharp, and trivially provides $\frac{3}{8}$ as a bound for the quantity in the conjecture.
- It took 25 years for this bound to be improved. In 2017, Boppana and Holzman achieved the best known bound for the problem. That is,

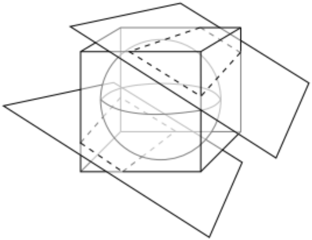
$$P[|\alpha \cdot x| \leq 1] > \frac{13}{32}.$$

Geometric Approach

Conjecture

Consider the unit n-ball and the cube around it. For each pair of parallel supporting hyperplanes of the ball, at least half of the cube's vertices lie in between (or on them).





Geometric Approach

Proposition

If $x, y \in \{-1, 1\}^n$ agree in all coordinates except one, i.e., belong to the same facet of the cube, then $\alpha \cdot x \leq 1$ or $\alpha \cdot y \leq 1$ (or both).

Corollary

Any supporting hyperplane of the ball can separate, i.e., $\alpha \cdot x > 1$, at most half the vertices of every facet of the cube from the sphere.

- This would have been the end of it, if a hyperplane does not cross the interior of at least one facet of the cube.
- For 5 or higher dimensions there are supporting hyperplanes that cross all facets.

Probabilistic Proof Strategy

Without loss of generality, we can assume that

- all coordinates of α are positive,
- $\alpha_1 + \alpha_2 \leq 1$

Then,

- define partial sums and stopping time:

$$S = \sum_{i=1}^{n} \alpha_i x_i, \; X_t = \sum_{i=1}^{t} \alpha_i x_i, \; Y_t = \sum_{i=t+1}^{n} \alpha_i x_i$$

$$T = \inf\{t > 0 : |X_t| > 1 - \alpha_{t+1} \text{ or } t = n-1\}$$

- and we want $P[|S| \leq 1] \geq \frac{1}{2}$.

Probabilistic Proof Strategy

- Instead of bounding $P[|S| \leq 1]$, bound $P[|S| \leq 1|T]$ for the **worst case** of T.
- Because $\alpha_1 + \alpha_2 \leq 1 \to T \geq 2$ (very important)
- Now, due to our assumptions, we obtain

$$\begin{aligned} P[|S| \leq 1|T] &\geq P[0 \leq Y_T \leq 2 - \alpha_{T+1}] \\ &\geq \frac{1}{2}\left(1 - P[|Y_T| > 2 - \alpha_{T+1}]\right). \end{aligned}$$

- Additionally,

$$E[Y_T^2] = \alpha_{T+1}^2 + \cdots + \alpha_n^2 = 1 - \alpha_1^2 - \cdots - \alpha_T^2 \leq 1 - T\alpha_{T+1}^2.$$

Tail Inequalities

Finally, we obtain the following two bounds

- Chernoff bound:

$$P[|S| \leq 1|T] \geq \frac{1}{2}\left(1 - e^{-\frac{2}{1-(T+1)\alpha_{T+1}^2}} - e^{-\frac{2(1-x)}{1-(T+1)\alpha_{T+1}^2}}\right)$$
$$\geq 0.358$$

- Fourth moment Markov's inequality:

$$P[|S| \leq 1|T] \geq \frac{1}{2}\left(1 - \frac{3(1 - T\alpha_{T+1}^2)^2}{(2 - \alpha_{T+1})^4}\right) \geq 0.377$$

Berry-Essen Theorem

Theorem

Let $X_1, \ldots, X_n$ be independent random variables with $E[X_i] = 0$ and $Var[X_i] = \sigma_i^2$, and assume $\sum_{i=1}^n \sigma_i^2 = 1$. Let $S = \sum_{i=1}^n X_i$ and let $Z \sim N(0,1)$ be a standard Gaussian. Then for all $u \in \mathbb{R}$,

$$|P[S \leq u] - P[Z \leq u]| \leq c\gamma,$$

where $\gamma = \sum_{i=1}^n \|X_i\|_3^3$ and c is a universal constant. ($c = 0.56$ is acceptable)

Remark

The quantity γ can also be replaced by the regularity of the X_i's, if any. That is, if $|X_i| \leq \epsilon$ for all $i \in [n]$, then

$$\gamma = \sum_{i=1}^n \|X_i\|_3^3 \leq \epsilon \sum_{i=1}^n \|X_i\|_2^2 = \epsilon.$$

Berry-Essen Theorem

By applying directly, we obtain

$$P[|\alpha \cdot x| \leq 1] \geq 0.68 - 1.12\gamma.$$

Then,

- Assume regularity. We can achieve $\frac{1}{2}$ for $|a_i| \leq 0.16$.
- Bound the error,

$$1.12\gamma \leq 0.61 + 1.4 \cdot 2^{-\frac{3n}{2}}.$$

Both approaches are very loose.

Conjecture

Let $\alpha \in \mathbb{R}^n$ with $\|\alpha\|_2 = 1$, and a function $f_\alpha : \{-1, 1\}^n \to \mathbb{F}_2$ defined by

$$f_\alpha(x) = \mathbb{1}_{\alpha \cdot x > 1}.$$

Then,

$$\|f_\alpha\|_1 \leq \frac{1}{4}.$$

Define the influence of a boolean function

$$Inf_i[f] = P[f(x) \neq f(x^{\oplus i})].$$

Try to bound the total influence $I[f] = \sum_{i=1}^{n} Inf_i[f]$ and then use the edge-isoperimetric inequality

$$I[f] \geq 2\|f_\alpha\|_1 \log_2 \left(\frac{1}{\|f_\alpha\|_1} \right).$$

Future directions

- Make remarks about the distribution of T.
- It would seem interesting to find some sequence of vectors $\{b_k\}_{k\in\mathbb{N}}$ satisfying:
  - $b_0 = \alpha$,
  - $P[|b_k \cdot x| \leq 1] \geq P[|b_{k+1} \cdot x| \leq 1]$,
  - $\lim_{k\to\infty} b_k = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right)$.
- A bound that depends on the minimum distance of the vector α from the worst case one. For some $L \leq \frac{1}{2}$ and $c > 0$,

$$P[|\alpha \cdot x| \leq 1] \geq L + c \cdot \min \left\| \alpha - \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right) \right\|_2$$
$$= L + c\sqrt{2 - \sqrt{2}(\alpha_1 + \alpha_2)},$$

where α_1, α_2 are the two greatest coordinates.

Thank you very much!