The Representation of Lie Colour Algebras and its connection with the Brauer algebra

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2. We call $(\rho|_W, W)$ a sub-representation if W is A-invariant subspace of V.
3. A representation (ρ, V) of A is called irreducible if there is no nontrivial proper sub-representation.

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2. The Young diagram of λ, $Y(\lambda)$, is a collection of boxes, arranged in left-justified rows, with λ_i boxes in row i.

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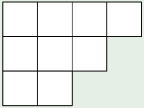
Example

$\lambda = (4, 3, 2)$. Then $Y(\lambda) =$



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$S_{\yng(5)}$ is the trivial representation of S_5.
$S_{\yng(1,1,1,1,1)}$ is the sign representation of S_5.

In representation theory, the **Schur-Weyl duality** relates the irreducible finite-dimensional representations of the general linear group GL_n and symmetric groups S_k

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Fact

The actions ρ_k and σ_k commute.

The Schur-Weyl duality

a. The action of S_k on $V^{\otimes k}$ generates $\mathrm{End}_{GL(n)}(V^{\otimes k})$.

b. The action of GL_n on $V^{\otimes k}$ generates $\mathrm{End}_{S_k}(V^{\otimes k})$.

By Schur-Weyl duality, we can decompose $V^{\otimes k}$ as the following:

$$V^{\otimes k} \cong \bigoplus_{Y(\lambda)} E^{Y(\lambda)} \otimes F^{Y(\lambda)}$$

where $Y(\lambda)$ is Young diagrams with k boxes and there are at most n rows.
Moreover, $E^{Y(\lambda)}$ is irreducible GL_n-representations and $F^{Y(\lambda)}$ is irreducible S_k-representations.

Question

Why the Schur-Weyl duality is correct?

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Key point

The actions of the S_k and the GL_n each generates the full centralizer of the action of the other.

Question

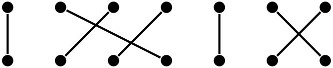
What if GL_n is replaced by some of its subgroups? For example the orthogonal group $O(n)$ and the symplectic group $Sp(2n)$. Does this theorem still hold?

Answer

The answer is No. But we can have a fix.

The solution is called the **Brauer algebra**.

An element in S_7 can be viewed by the following diagram:



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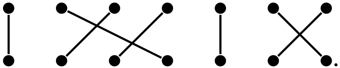

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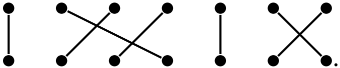

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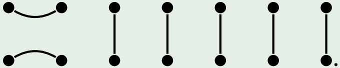
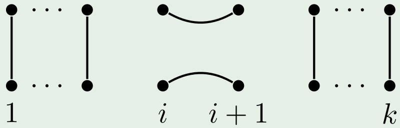
A horizonal edge



Generators

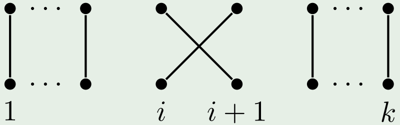
The Brauer algebra $B_k(\eta)$ is generated by the diagrams e_i and s_i for all $1 \leq i \leq k-1$, where

$e_i =$



and

$s_i =$



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$B_2(\eta)$

$$e_1 s_1 = s_1 e_1 = e_1, \qquad s_1^2 = 1, \qquad e_1^2 = \eta e_1$$

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The Brauer algebra is NOT always semisimple. For example $B_2(0)$ is not semisimple. However $B_2(\eta)$ is semisimple for all $\eta \neq 0$.

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Matrix realization

$$\iota = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, E_1 = \begin{pmatrix} \eta & & \\ & 0 & \\ & & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

It is straightforward to check that

$$E_1 S_1 = S_1 E_1 = E_1, \qquad S_1^2 = \iota, \qquad E_1^2 = \eta E_1.$$

Theorem 1

a. The action of $B_k(n)$ on $V^{\otimes k}$ generates $\mathrm{End}_{O(n)}(V^{\otimes k})$.

b. The action of $O(n)$ on $V^{\otimes k}$ generates $\mathrm{End}_{B_k(n)}V^{\otimes k}$.

Theorem 2

a. The action of $B_k(-2n)$ on $V^{\otimes k}$ generates $\mathrm{End}_{Sp(2n)}(V^{\otimes k})$.

b. The action of $Sp(2n)$ on $V^{\otimes k}$ generates $\mathrm{End}_{B_k(-2n)}V^{\otimes k}$.

Questions

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Answers

1. Yes, we can form $SpO(n|2m)$
2. Yes, the corresponding Lie algebra is $\mathfrak{spo}(n|2m)$

Even better

We are able to construct a more generalized term called the orthosymplectic Lie colour algebra $\mathfrak{spo}(V, \beta)$ and we can get a Schur-Weyl duality-like decomposition.

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$$V = \bigoplus_{a \in G} V_a.$$

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Definition

β is a bilinear map from $G \times G$ to $\mathbb{F}^*$.

Definition

Given a symmetric bicharacter β on an abelian group G. A *Lie colour algebra* over a field $\mathbb{F}$ is a G-graded vector space with the following grading system

$$\mathfrak{g} = \bigoplus_{a \in G} \mathfrak{g}_a,$$

equipped with a binary operation map, the *Lie colour bracket*

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

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2. $[x, y] = -\beta(b, a)[y, x]$, for all $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b$,
3. $\beta(a, c)[x, [y, z]] + \beta(b, a)[y, [z, x]] + \beta(c, b)[z, [x, y]] = 0$, for all $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b$ and $z \in \mathfrak{g}_c$.

Fact

Lie colour algebra is a more general concepts of Lie algebra and Lie supalgebra.

Definition

Let V be a G-graded vector space. Let $\langle , \rangle$ be a non-degenerate bilinear form on V. The orthosymplectic Lie colour algebra is the Lie colour subalgebra

$$\mathfrak{spo}(V,\beta) = \bigoplus_{a\in G} \mathfrak{spo}(V,\beta)_a,$$

where

$$\mathfrak{spo}(V,\beta)_a = \{x \in \mathfrak{gl}(V,\beta)_a \mid \langle xw, v\rangle + \beta(b,a)\langle w, xv\rangle = 0, \\ \forall w \in V_b,\ v \in V\}.$$

Commuting Actions

We have an algebra homomorphism from

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Georgia Benkart et.al. Tensor product representations for orthosymplectic Lie superalgebras, 1998

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Theorem

Let $V^{\otimes k}$ be be tensor product of standard $\mathfrak{spo}(V,\beta)$-modules. Let λ be a partition, and let λ' be the transpose of λ. Then

$$U^\lambda := V^{\otimes k} e_{\lambda'}$$

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Let $|n-m| > k$. Then as an $\mathfrak{spo}(V,\beta) \times B_k(n-m)$ module, we have

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2. We do not know if U^{λ} is always irreducible.

Thank you!