Rational approximations on algebraic varieties

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June 15, 2018

Overview

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How well can we do this with respect to the size of q?
i.e. $\left| x - \frac{p}{q} \right| \leq \phi(q)$?

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If we know that $\left|\sqrt{2} - \frac{p}{q}\right| \leq \frac{1}{q^2}$. Then $\sqrt{2}$ is irrational.

Approximations in $\mathbb{R}$

Theorem [Dirichlet (1842)]. Let $x \in \mathbb{R}$, $x \notin \mathbb{Q}$. Then there are infinitely many rational numbers $\frac{p}{q}$ such that $|x - \frac{p}{q}| \leq \frac{1}{q^2}$.

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Consider the equation $x^2 - dy^2 = 1$, where d is not a perfect square. Then any nontrivial solution (p, q) of this equation is a solution of $|\sqrt{d} - \frac{p}{q}| \leq \frac{1}{q^2}$

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This is how Liouville proved that, for instance, $\sum_{k=0}^{\infty} 10^{-k!}$ is a transcendental number.

Roth's Theorem

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Roth was awarded a fields medal for the proof of this result!

Approximations on Algebraic Varieties

Given a projective variety X defined over $\mathbb{Q}$ (zeros of homogeneous polynomials with coefficients in $\mathbb{Q}$) and a point $x \in X(\overline{\mathbb{Q}})$ we want to make sense of approximating rational points on X.

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Example

Given the variety $X = \{x^2 - 2y^2 = 1\}$ and a point $(\sqrt{2}, 1) \in X(\overline{\mathbb{Q}})$ how well can we approximate $(\sqrt{2}, 1)$ by rational points in X?

Approximations on Algebraic Varieties

points $\frac{p}{q} \in \mathbb{Q}$ such that $q\left|x - \frac{p}{q}\right|^{\frac{1}{2}-\epsilon} \leq 1$

↓

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3. Exponent γ

Height functions

Let X be a variety over $\mathbb{Q}$. A **height function** is a function $H : X(\mathbb{Q}) \to \mathbb{R}$ that measures the complexity of each point.

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- For any real number B, the set $\{x \in X(\mathbb{Q}) : \; H(x) \leq B\}$ is finite.
- Geometric relations lead to height relations.

Height functions on $\mathbb{P}^n$

Definition: Let $x = [x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{Q})$. Such that $x_i \in \mathbb{Z}$ for all i and $gcd(x_0, \cdots, x_n) = 1$. We define the height of x as

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Notice that

- For any real number B, the set $\{x \in \mathbb{P}^n(\mathbb{Q}) : \ H(x) \leq B\}$ is finite.
- A rational point $\frac{p}{q}$ can be seen as the point $[p : q] \in \mathbb{P}^1(\mathbb{Q})$. Then $H([p : q]) = \max(|p|, |q|)$

Height functions on projective varieties

Let X be a projective variety with an embedding $\phi : X \hookrightarrow \mathbb{P}^n$.
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More intrinsically, a projective embedding corresponds to a very ample divisor D. So for each very ample divisor D we can choose an embedding $\phi_D : X \hookrightarrow \mathbb{P}^n$ and get a height function $h_D : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ such that $h_D(x) = H(\phi_D(x))$

Distance

Let X be a projective variety over Spec(k). Fix a place v_0 on k and an extension v to $\bar{k}$. Choose and embedding $X \hookrightarrow \mathbb{P}^n$. Let $x = [x_0 : \cdots : x_n],\ y = [y_0 : \cdots : y_n] \in \mathbb{P}^n(\bar{k})$.

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If v archimedean:

$$d_v(x, y) = \left(1 - \frac{|\sum_{i=1}^n x_i \overline{y_i}|^2}{(\sum_{i=1}^n |x_i|^2)(\sum_{i=1}^n |y_i|^2)}\right)^{[k_v:\mathbb{R}]/2}$$

Distance

Let X be a projective variety over $\mathrm{Spec}(k)$. Fix a place υ_0 on k and an extension υ to $\bar{k}$. Choose and embedding $X \hookrightarrow \mathbb{P}^n$. Let $x = [x_0 : \cdots : x_n]$, $y = [y_0 : \cdots : y_n] \in \mathbb{P}^n(\bar{k})$.

If υ archimedean:

$$d_\upsilon(x, y) = \left(1 - \frac{|\sum_{i=1}^n x_i \overline{y_i}|^2}{(\sum_{i=1}^n |x_i|^2)(\sum_{i=1}^n |y_i|^2)}\right)^{[k_\upsilon:\mathbb{R}]/2}$$

If υ non-archimedean:

$$d_\upsilon(x, y) = \frac{\max_{0 \le i < j \le n} ||x_i y_j - x_j y_i||_\upsilon}{(\max_{0 \le i \le n} ||x_i||_\upsilon)(\max_{0 \le j \le n} ||y_j||_\upsilon)}$$

Approximation constant α

Let X be a projective variety, $x \in X(\overline{\mathbb{Q}})$, D be a divisor on X. For any sequence $\{x_i\} \subseteq X(\mathbb{Q})$ of distinct points such that $d_v(x, x_i) \to 0$ we define

$$\alpha_x(\{x_i\}, D) := \inf \left\{ \gamma \in \mathbb{R} : \begin{array}{l} d_v(x, x_i)^\gamma H_D(x_i) \\ \text{is bounded} \end{array} \right\},$$

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If no such a sequence exists, set $\alpha_x(D) = \infty$.

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For $X = \mathbb{P}^1(\mathbb{Q})$, $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$, and H a hyperplane:

$$\begin{aligned}
\text{Liouville} &: \quad \alpha_x(H) \geq \frac{1}{d}, \text{ where } d = [\mathbb{Q}(x) : Q], \\
\text{Dirichlet} &: \quad \alpha_x(H) \leq \frac{1}{2}, \ x \in \mathbb{R} \cap \overline{\mathbb{Q}}, \ x \notin \mathbb{Q} \\
\text{Roth} &: \quad \alpha_x(H) \geq \frac{1}{2}, \ x \in \overline{\mathbb{Q}}
\end{aligned}$$

Some results

Theorem: Let $x \in \mathbb{P}^n(\mathbb{Q})$ and H be a divisor of a hyperplane in $\mathbb{P}^n$, then $\alpha_x(H) = 1$. Approximations can be taken on any line through x.

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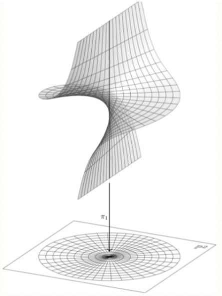
Theorem: Let X be the blowup of $\mathbb{P}^2$ at m k-rational points in general position for $m \in \{1, 2, 3, 4, 5, 6\}$. For any point $x \in X(k)$ and any ample divisor D, $\alpha_D(x)$ exist and there is a curve that contains the best approximations.



Theorem: For all Hirzebruch surfaces $\mathcal{H}_n = \{([x_0 : x_1], [y_0 : y_1 : y_2]) : \; x_0^n y_1 = x_1^n y_2\} \subset \mathbb{P}^1 \times \mathbb{P}^2$, $n \geq 0$, any point $x \in H_n(k)$, and any ample divisor D, $\alpha_D(x)$ exist and there is a curve that contains the best approximations.

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References

David McKinnon.

A conjecture on rational approximations to rational points.

J. Algebraic Geom., 16(2):257–303, 2007.

David McKinnon and Mike Roth.

An analogue of Liouville's theorem and an application to cubic surfaces.

Eur. J. Math., 2(4):929–959, 2016.

Marc Hindry and Joseph H. Silverman.

Diophantine geometry, volume 201 of *Graduate Texts in Mathematics*.

Springer-Verlag, New York, 2000.

An introduction.

Thank You!