Induction Relations in Symmetric Groups and Jucys-Murphy Elements

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June 16, 2018

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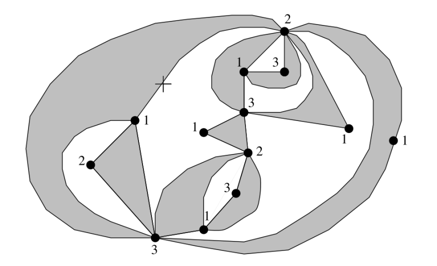
Can we take advantage of the natural inclusions

$$\varnothing \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{S}_3 \subseteq \cdots$$

to count?

Things you might like

Ramified covering of Riemann spheres and hypermaps are encoded by $(\sigma_1, \ldots, \sigma_m)$ with $\sigma_i \in \mathcal{S}_n$.



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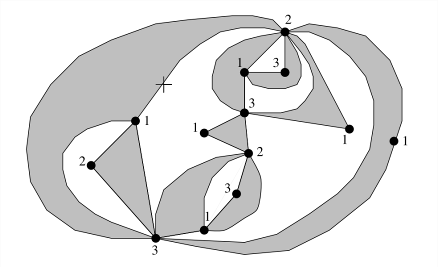

- *transitive* if the subgroup $\langle \sigma_1, \ldots, \sigma_m \rangle$ acts transitively on $\{1, \ldots, n\}$. Captures connectedness of encoded objects.
- *genus* g is given by Riemann-Hurwitz formula or Euler's formula. They agree.

$1, 6, 22, 174, 1479, 16808, \ldots$

Let $S_1, \ldots, S_m$ be non-empty subsets of $\mathcal{S}_n$. How many ways can one write a permutation $\sigma \in \mathcal{S}_n$ as

$$\sigma = \sigma_1 \cdots \sigma_m,$$

with $\sigma_k \in S_k$ for $k \geq 1$.

The answer is $|S_1| \cdots |S_m|/n!$ if some $S_i = \mathcal{S}_n$. But in general this is not easy.

Algebra is always the answer

The *group algebra* $\mathbb{C}[\mathcal{S}_n]$ is the vector space spanned by $\mathcal{S}_n$ as formal symbols with multiplication

$$\left(\sum_{\sigma\in\mathcal{S}_n} a_\sigma \boldsymbol{\sigma}\right)\left(\sum_{\tau\in\mathcal{S}_n} b_\tau \boldsymbol{\tau}\right) = \sum_{\sigma,\tau\in\mathcal{S}_n} a_\sigma b_\tau \boldsymbol{\sigma}\boldsymbol{\tau}, \quad a_\sigma, b_\tau \in \mathbb{C}.$$

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Generating series:

$$\Phi_n^{S_1,\dots,S_m} = \left(\sum_{\sigma_1\in S_1} \boldsymbol{\sigma_1}\right)\cdots\left(\sum_{\sigma_m\in S_m} \boldsymbol{\sigma_m}\right) = \sum_{\sigma_i\in S_i} \boldsymbol{\sigma_1}\cdots\boldsymbol{\sigma_m}.$$

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Answer to factorization problem is simply

$$[\boldsymbol{\sigma}]\Phi_n^{S_1,\ldots,S_m}.$$

Centre is better

A natural basis of $Z(\mathbb{C}[\mathcal{S}_n])$:

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Theorem (Jucys '74, Murphy '81)
If $f \in \mathrm{Sym}_n$ is a symmetric polynomials in n variables, then $f(J_1, J_2, \ldots, J_n) \in \mathcal{Z}_n$ where

$$J_k = (1, k) + (2, k) + \cdots + (k-1, k), \quad k = 2, \ldots, n.$$

Useful tools

Well-known symmetric functions

$$\text{power sum: } p_k = x_1^k + x_2^k + \cdots$$
$$\text{complete: } h_k = \sum_{\substack{a_1, a_2, \cdots \geq 0 \\ a_1 + a_2 + \cdots = k}} x_1^{a_1} x_2^{a_2} \cdots$$

Multiplicative bases: $p_\alpha = p_{\alpha_1} p_{\alpha_2} \cdots$ where $\alpha \vdash n$ is a partition.

Symmetric polynomials are specializations:

$$f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, 0, 0, \ldots).$$

Inner product: $\langle p_\alpha, p_\beta \rangle = \frac{n!}{|\mathcal{C}_\alpha|} \delta_{\alpha\beta}$.

Generating series again

If S_i's are unions of conjugacy classes, then $\Phi_n^{S_1,\ldots,S_m}$ is central! Moreover,

$$\Phi_n^{S_1,\ldots,S_m}(\mathbf{p}) = \mathrm{ch}^n \Phi_n^{S_1,\ldots,S_m} = \frac{1}{n!}\sum_{\alpha \vdash n} [\mathcal{C}_\alpha]\Phi_n^{S_1,\ldots,S_m} |\mathcal{C}_\alpha| p_\alpha.$$

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Let $f_1,\ldots,f_m \in \mathrm{Sym}$. Define

$$\Phi^{f_1,\ldots,f_m}(z,\mathbf{p}) = \sum_{n\geq 0} z^n \mathrm{ch}^n \prod_{k=1}^m f_k(J_1,\ldots,J_n)$$
$$= \sum_{n\geq 0}\frac{z^n}{n!}\sum_{\alpha\vdash n}[\mathcal{C}_\alpha]\prod_{k=1}^m f_k(J_1,\ldots,J_n)|\mathcal{C}_\alpha| p_\alpha$$

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Transitive factorizations: $\Psi = \log\Phi$.

Hurwitz numbers

Transitive factorization into transpositions $(a_1 b_1) \cdots (a_k b_k) = \sigma$.

$$\Psi^{e^{p_1}}(t, z, \mathbf{p}) = \log \sum_{n \geq 0} \frac{z^n}{n!} \mathrm{ch}^n \left(\sum_{\alpha \vdash n} \exp(tJ_1 + \cdots + tJ_n) \right).$$

Studied by Goulden and Jackson in 1997. Genus 0 count

$$(n + \ell(\alpha) - 2)! n^{\ell(\alpha)-3} \prod_{i=1}^{\ell(\alpha)} \frac{\alpha_i^{\alpha_i}}{(\alpha_i - 1)!}.$$

Hypermap number

No restrictions

$$\Psi^{E^m}(t, z, \mathbf{p}) = \log \sum_{n \geq 0} \frac{z^n}{n!} \mathrm{ch}^n \left(\sum_{m \geq 1} \mathcal{S}_n^m t^m \right).$$

Planar case studied by Bousquet-Mélou and Schaeffer in 2000 using bijection with a certain family of decorated trees.

$$m \frac{((m-1)n-1)!}{((m-1)n-\ell(\alpha)+2)!} \prod_{i=1}^{\ell(\alpha)} \binom{m\alpha_j - 1}{\alpha_j}.$$

Monotone Hurwitz number

A factorization $(a_1 b_1) \cdots (a_k b_k) = \sigma$ into transpositions is *monotone* if $a_i < b_i$ and $b_1 \leq \cdots \leq b_k$.

$$\Psi^H(t, z, \mathbf{p}) = \log \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\alpha \vdash n} \mathrm{ch}^n \left(\sum_{k \geq 1} t^k (h_k(J_1, \ldots, J_n)) \right).$$

Studied by Guay-Paquet, Goulden and Novak in 2013. Genus 0 count

$$\frac{(2n + \ell(\alpha) - 3)!}{(2n)!} \prod_{j=1}^{\ell(\alpha)} \binom{2\alpha_j}{\alpha_j}.$$

Joins and Cuts

The combinatorial behaviour of $(ab) \cdot \sigma$ in $\mathcal{S}_n$ boils down to joins and cuts:

- *cut* if a, b on different cycles.
- *join* if a, b on the same cycle.

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Algebraic tool:

$$\Delta = \frac{1}{2} \sum_{i,j \geq 1} p_i p_j p_{i+j}^{\perp} + p_{i+j} p_i^{\perp} p_j^{\perp},$$

where $\langle p_i^{\perp} p_\alpha, p_\beta \rangle = \langle p_\alpha, p_i p_\beta \rangle$. In fact we know

$$p_i^{\perp} = i \frac{\partial}{\partial p_i}.$$

Operators

Define $\mathcal{U}_0 = p_1$ and $\mathcal{D}_0 = p_1^\perp$ and

$$\mathcal{U}_k = [\Delta, \mathcal{U}_{k-1}] \quad \text{and} \quad \mathcal{D}_k = [\mathcal{U}_{k-1}, \Delta], \quad k \geq 1.$$

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Note $\mathcal{U}_0 s_\lambda = p_1 s_\lambda$ is the Murnaghan-Nakayama rule. Moreover

$$\mathcal{U}_k s_\lambda = \sum_{\mu = \lambda + \square} c(\square)^k s_\mu \quad \text{and} \quad \mathcal{D}_k s_\lambda = \sum_{\mu = \lambda - \square} c(\square)^k s_\mu.$$

On power sums

Suppose $\alpha \vdash n$. Then

$$\mathcal{U}_k p_\alpha = \sum_{\tau \in J^k_{n+1}} p_{\mathrm{cyc}(\tau)}.$$

for any $\sigma \in \mathcal{C}_\alpha$.

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Some expressions:

$$\mathcal{U}_1 = \sum_{i \geq 1} p_{i+1} p_i^\perp,$$

$$\mathcal{U}_2 = \sum_{i,j \geq 1} \left(p_i p_j p_{i+j-1}^\perp + p_{i+j+1} p_i^\perp p_j^\perp \right)$$

$$\mathcal{D}_1 = \sum_{i \geq 1} p_i p_{i+1}^\perp,$$

$$\mathcal{D}_2 = \sum_{i,j \geq 1} \left(p_i p_j p_{i+j+1}^\perp + p_{i+j-1} p_i^\perp p_j^\perp \right).$$

A Recurrence

PDE for (not necessarily connected) monotone Hurwitz generating series $\Phi^H = \Phi^H(t, z, \mathbf{p})$ is

$$p_1^{\perp}\Phi^H = z\Phi^H + t^2 \sum_{i,j\geq 1} \left(p_i p_j p_{i+j+1}^{\perp} + p_{i+j-1} p_i^{\perp} p_j^{\perp} \right) \Phi^H$$

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This is due to Lassalle (2010). He applied $\mathcal{U}$ operators to obtain linear relations in central characters. He also used the symmetric function recurrence

$$h_k(x_1, \ldots, x_{n+1}) = h_k(x_1, ., x_n, 0) + x_{n+1} h_{k-1}(x_1, \ldots, x_{n+1}).$$

Combinatorial proof due to Féray.

One last generating series

Let $f(x)$ be a formal power series. The content series is

$$\Phi^f(t, z, \mathbf{p}) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\alpha \vdash n} \mathrm{ch}^n \prod_{k=1}^{n} f(tJ_k)$$
$$= \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\lambda \vdash n} \chi^{\lambda}_{(1^n)} \prod_{\square \in \lambda} f(tc(\square)).$$

Generalize all 3 aforementioned generating series.

It is the unique solution of

$$\sum_{i \geq 0} f_i t^i \mathcal{U}_i \Phi^f = \frac{\partial}{\partial z} \Phi^f$$

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Thank You!