(Minimal) Hypersurfaces in nearly G_2 manifolds

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What are G_2 structures ?

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Since a tangent space at every point of a 7-dimensional manifold can be identified with $\mathbb{R}^7$, so we make the following definition.

Definition

A cross product B on a Riemannian manifold (M^7, g) is an alternating map

$$B : TM \times TM \to TM$$

satisfying the following conditions

$$\begin{cases} g(B(v_1, v_2), v_i) = 0, & i = 1, 2 \\ \|B(v_1, v_2)\|^2 = \|v_1 \wedge v_2\|^2 \end{cases}$$

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- G_2 is one of the groups which appears on the Berger's (1955) classification of Riemannian holonomy groups.

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It's very surprising that the G_2 structure φ determines the metric g, the orientation and hence the volume form.

- The first local examples of metrics with holonomy G_2 were found by Robert Bryant in 1987.
- The first *complete* examples of G_2 holonomy metrics were found by Robert Bryant and Simon Salamon in 1989. They were non-compact examples.
- The first *compact* examples of G_2 holonomy metric were constructed by Dominic Joyce in 1994.
- There have been 3 more constructions of compact G_2 manifolds : Kovalev (2004), Corti-Haskins-Nordstörm-Paccini (2012) and Joyce-Karigiannis (2017). They all involve *gluing* certain areas and then solving a *highly* nonlinear PDE.
- Torsion free compact G_2 manifolds as important as they serve as "compactifications" in Witten's M-theory.

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- So, we can define an almost complex structure on a manifold M by having a "continuous" family of Js at T_pM for every point $p \in M$.
- There are manifolds which might not have *any* complex structure at all! for example the unit sphere S^4 *does not* admit an almost complex structure whereas S^2 and S^6 does.

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Lemma

The map J is an almost complex structure on M.

Proof.

Follows from properties of the cross product. □

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UPSHOT- Any hypersurface of a manifold with a G_2 structure has an almost complex structure.

We make the following

Definition

We say that a manifold (M, g, J) is a *nearly Kähler* manifold if

$$(\nabla_X J)(X) = 0$$

for all vector fields X on M.

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- On any oriented Riemannian manifold M^n, we have an operator $\star$ called the *Hodge star* which takes a differential k-form to a differential $(n-k)$-form, so it's sort of a "dual". **For example** consider $\mathbb{R}^3$ with coordinates $\{x, y, z\}$, then dx, dy, dz are a basis for the space of one forms. Then $\star dx = dy \wedge dz,\ \star dy = dz \wedge dx,\ \star dz = dx \wedge dy$

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Example The round sphere S^7 is a manifold with a nearly G_2 structure.

Now we saw that any hypersurface of a manifold with a G_2 structure has an almost complex structure J. We also saw that it might *not* be nearly Kähler, so a natural question is to look for conditions which guarantee that J is nearly Kähler.

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Theorem (D. '18)

Let M be an oriented hypersurface of a nearly G_2 manifold $(\overline{M}, \varphi)$. Then (M, g, J) is a nearly Kähler structure if and only if for all vector fields X in M

$$AX = \alpha X + \beta J(X)$$

where A is the shape operator of M in $\overline{M}$ which is a self-adjoint operator and $\alpha,\ \beta \in C^\infty(TM)$.

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- In fact, minimal submanifolds are critical points of the "Area functional". However, contrary to their names, they might not necessarily be minima, they are only critical points.
- Another definition of minimal submanifolds is this. Recall that the shape operator A is a self-adjoint operator, so we can diagonalize it. Suppose $\{a_1, a_2, ..., a_n\}$ are its eigenvalues. We define the *mean curvature* H by

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N is minimal if $H \equiv 0$.

- For compact, minimal hypersurfaces of constant scalar curvature R in the unit sphere S^{n+1}, the norm of the shape operator A satisfies $|A|^2 = n(n-1) - R$. So since R is constant so is $|A|^2$.

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Conjecture [Chern]

Consider the set $\{M^n \mid M \text{ is a hypersurface as above in } S^{n+1}\}$ and consider $|A|^2$ as a function on this set. Then the image of this function is a discrete set.

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  But there is no idea about the third value.

- Recall that S^7 has a nearly G_2 structure, so for Chern's conjecture, at least for the seven dimensional unit sphere, we can try to use this extra structure to extract some information about the third value of $|A|^2$.

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Theorem (D. '18)

Let M^6 be a compact minimal hypersurface of constant scalar curvature in the unit sphere S^7. If the shape operator A of M satisfies $|A|^2 > 6$, then there exists an eigenvalue $\lambda > 12$ of the Laplace operator on M such that $|A|^2 = \lambda - 6$.

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So the analysis about values of $|A|^2$ is now related to (hopefully) more tractable Laplacian operator.

Thank you for your attention!!