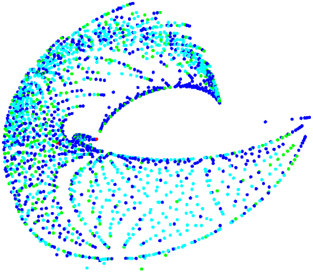
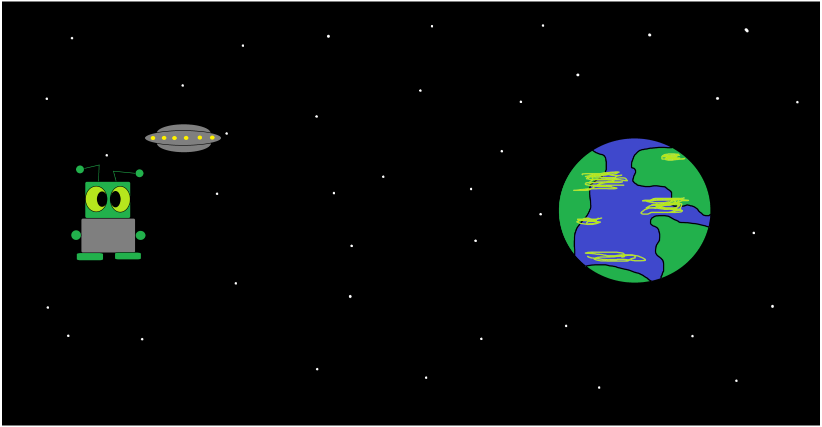
The Arithmetic of Algebraic Dynamics



Ehsaan Hossain
University of Waterloo

An alien satellite



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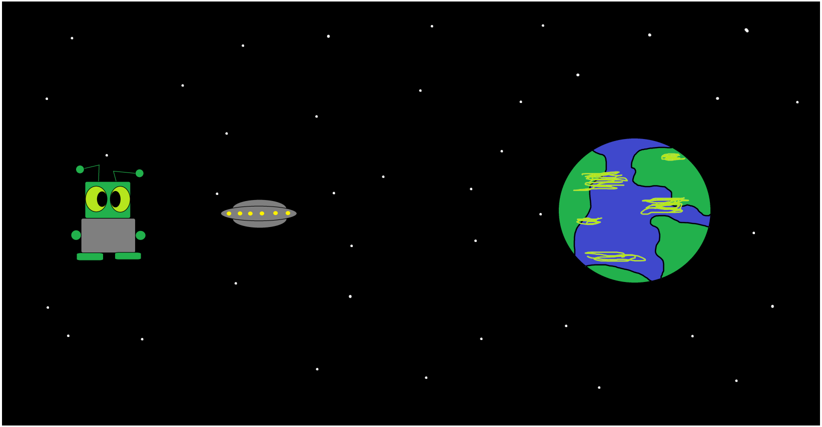

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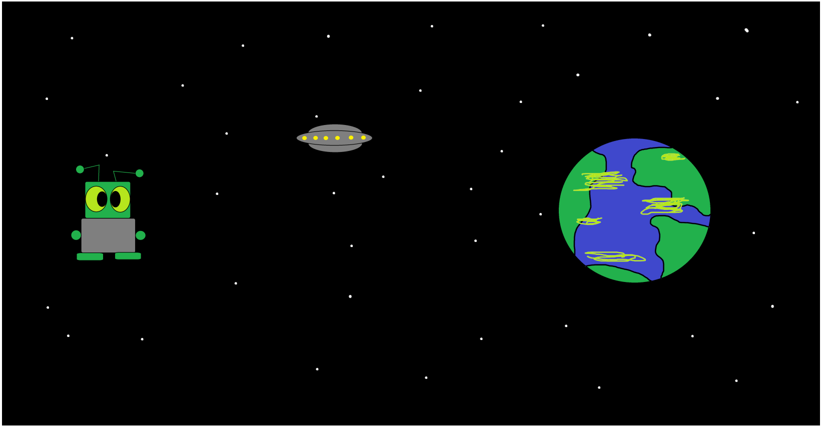

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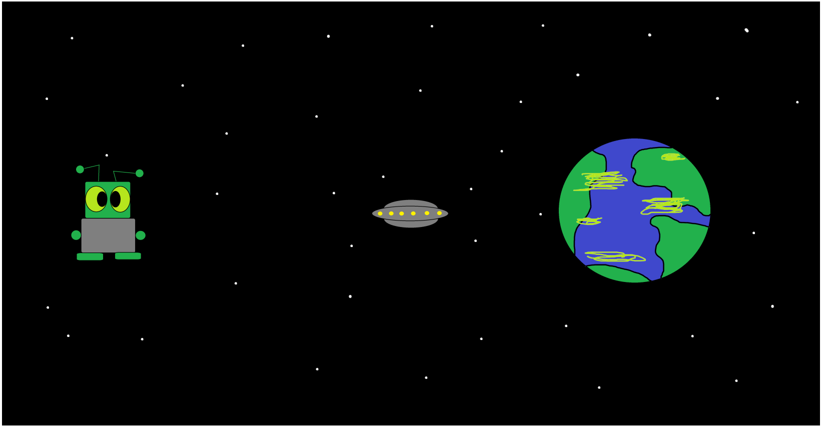

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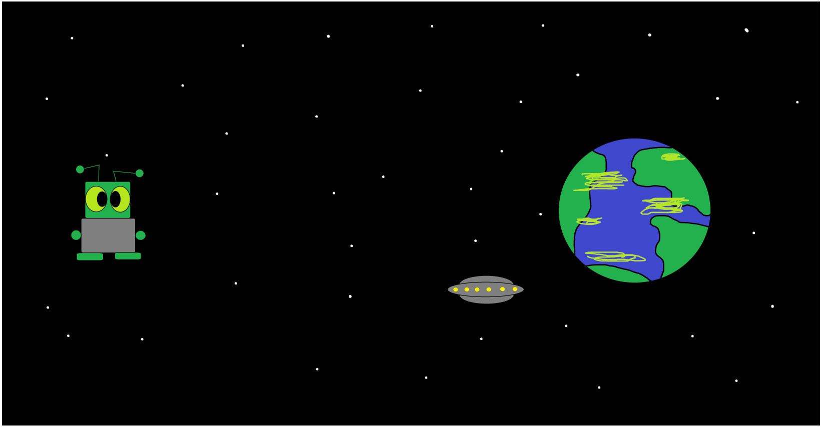

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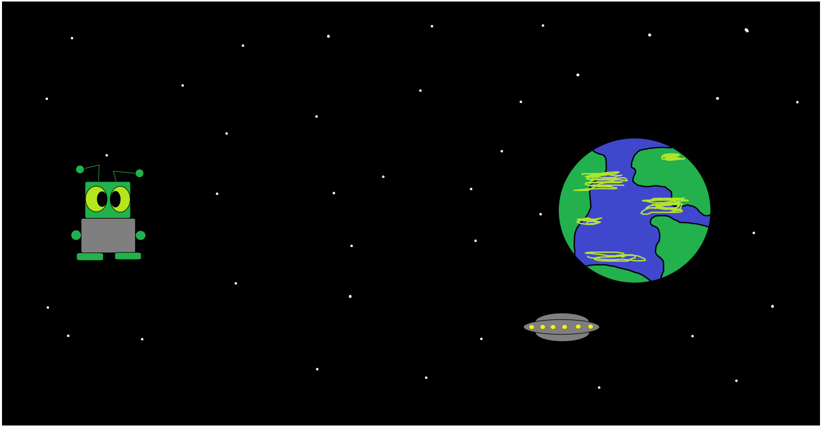

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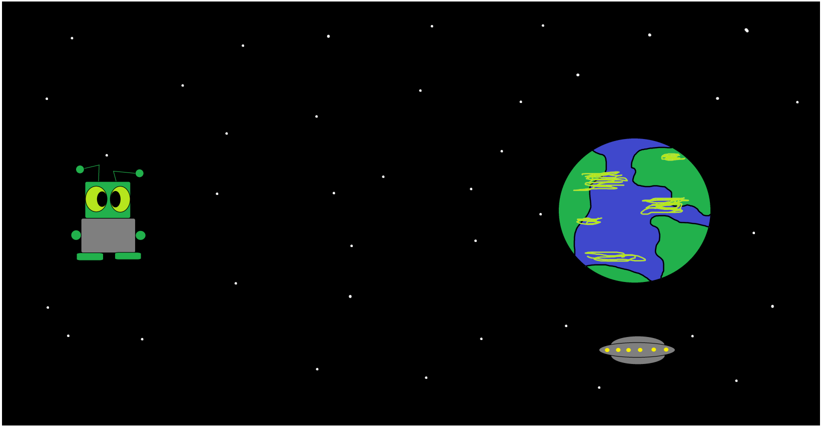

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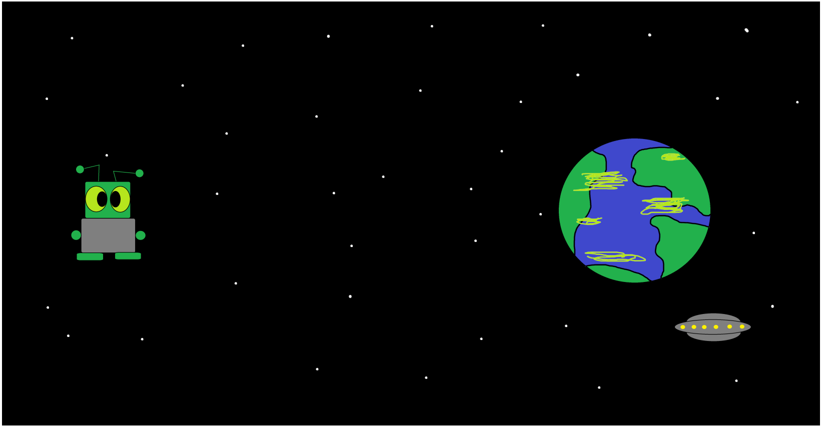

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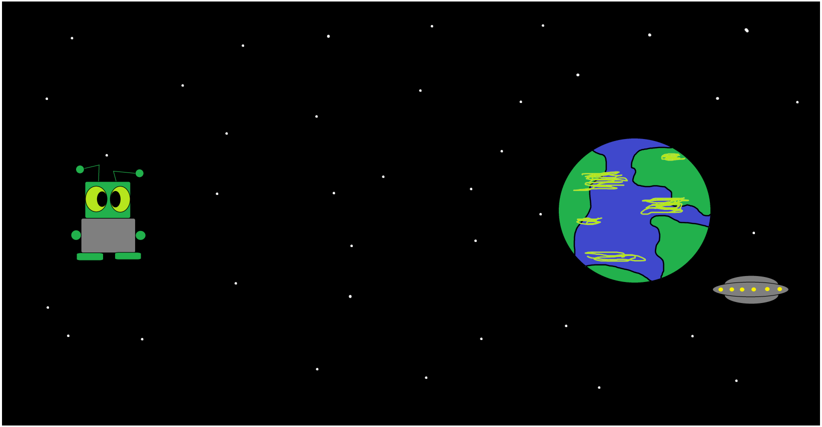

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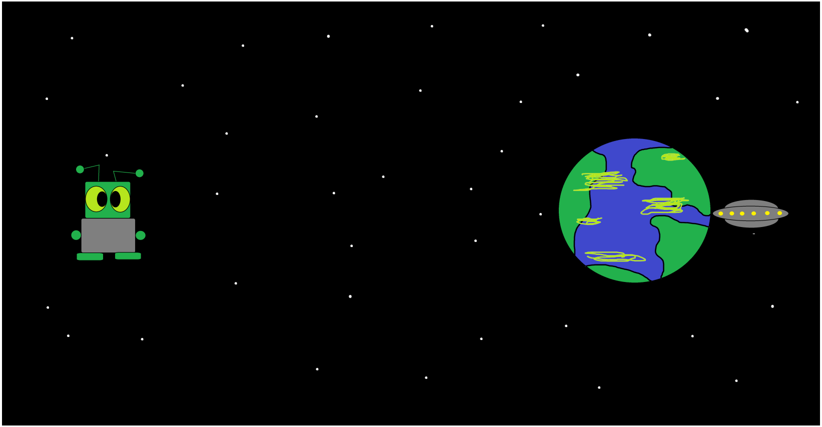

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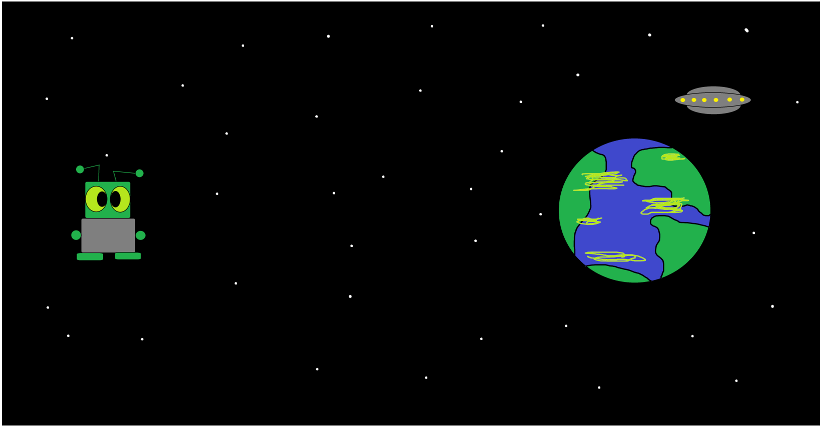

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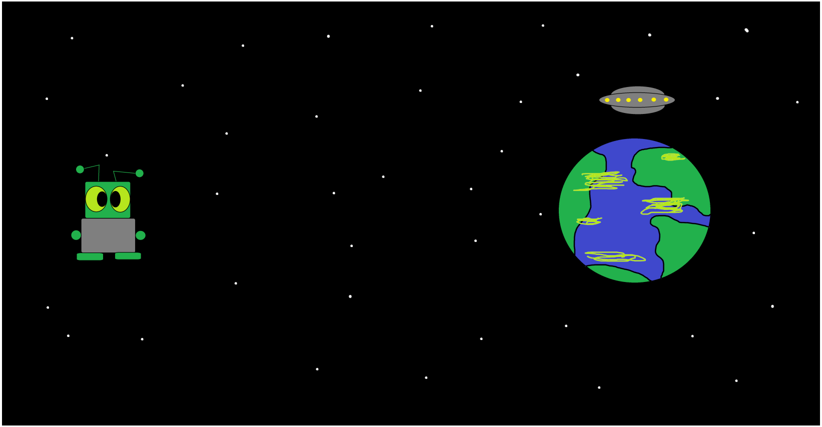

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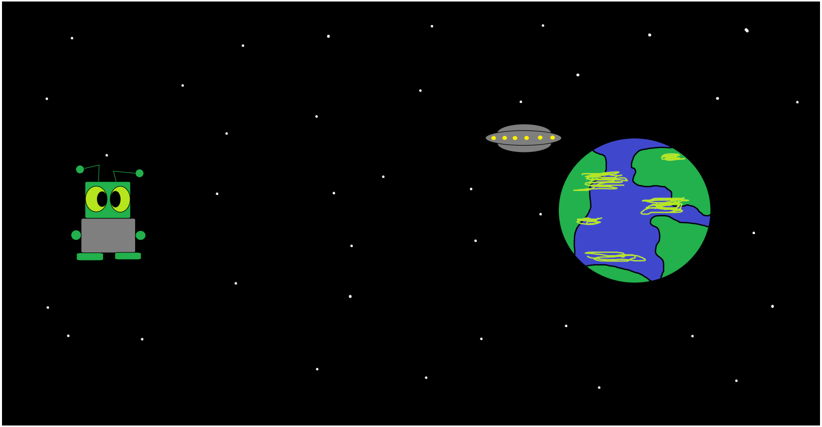

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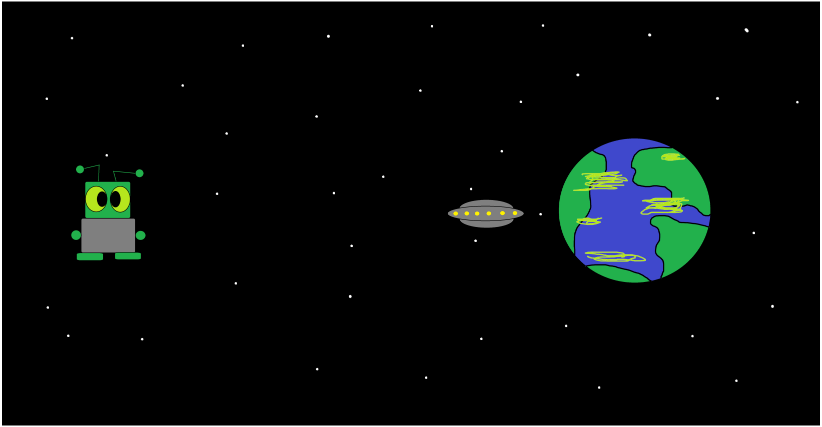

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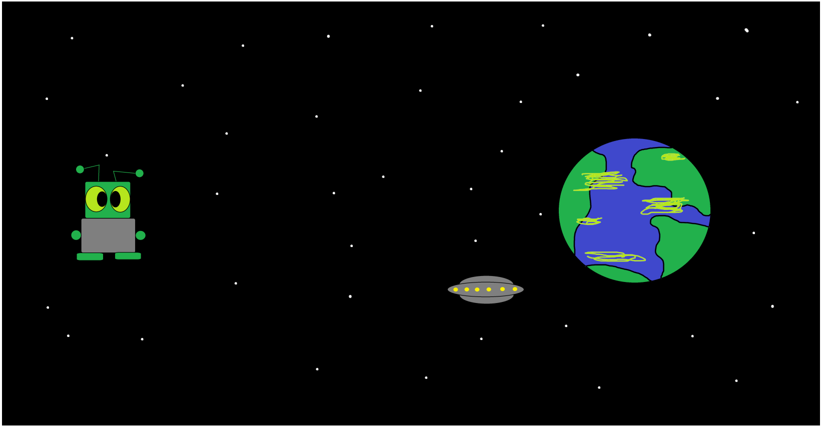

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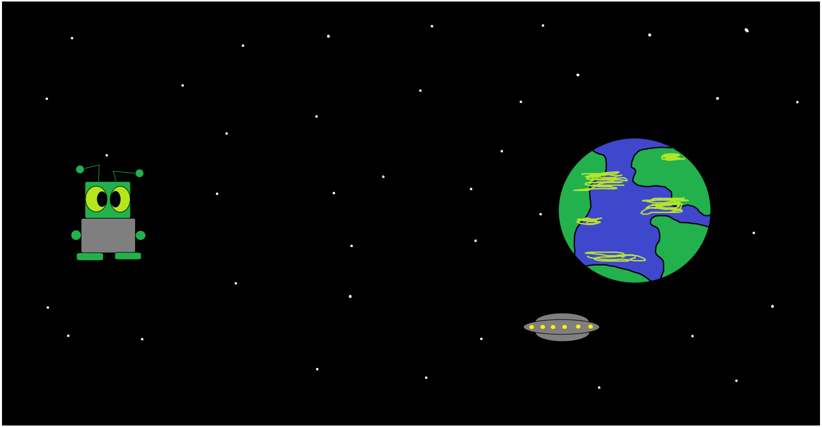

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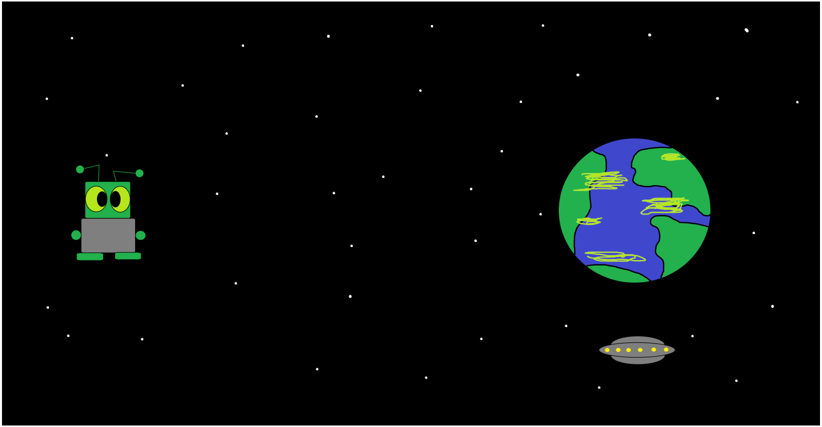

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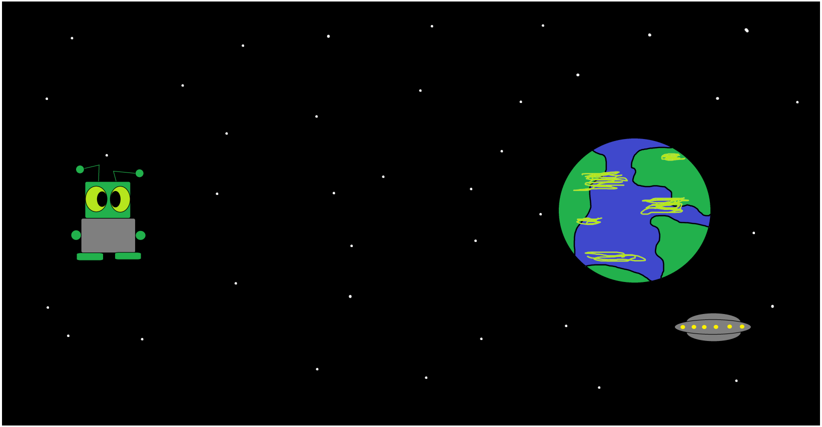

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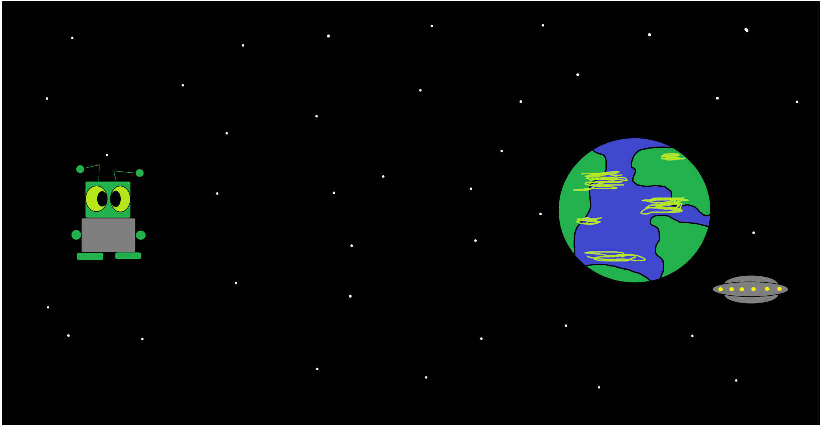

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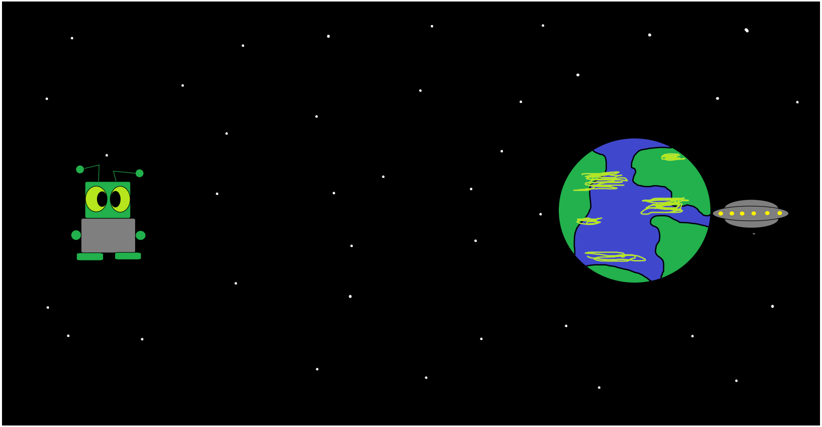

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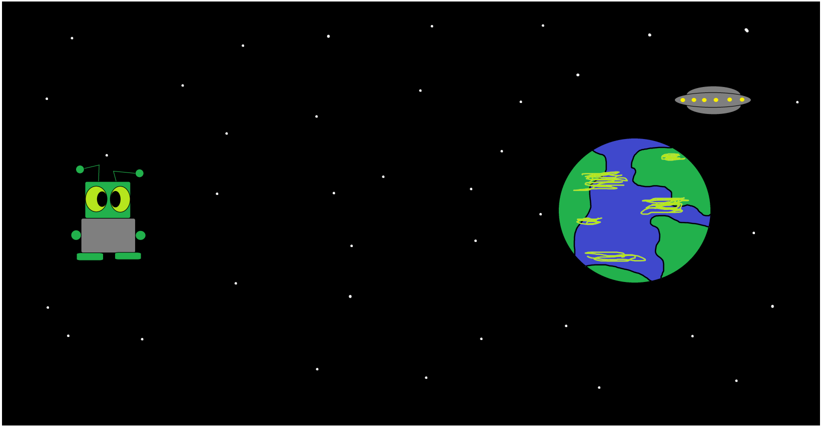

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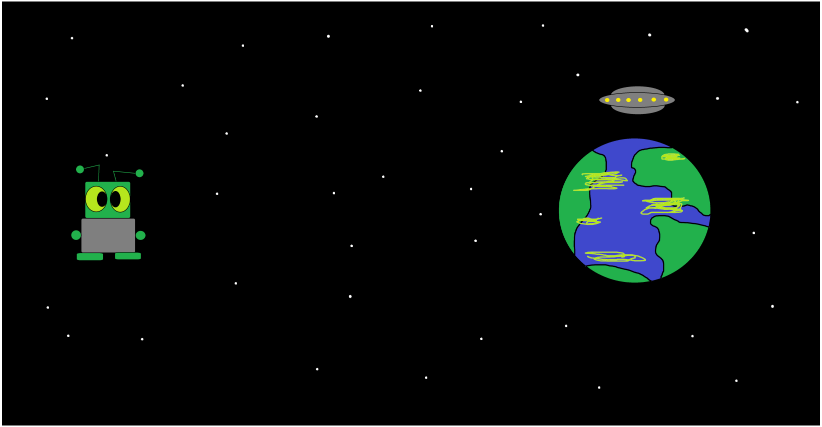

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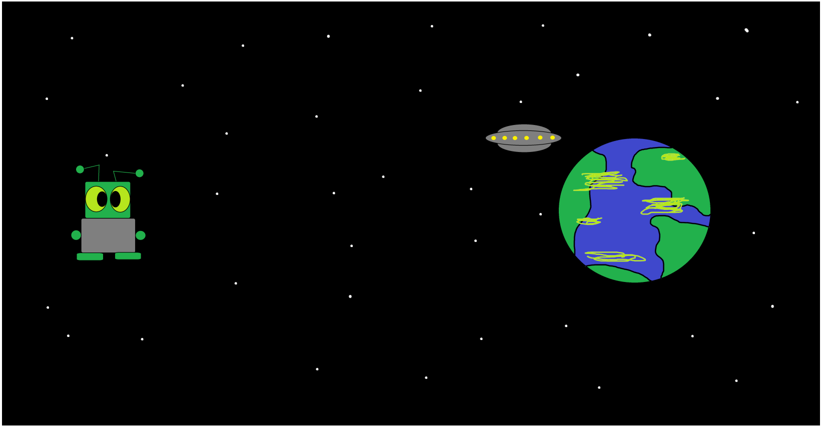

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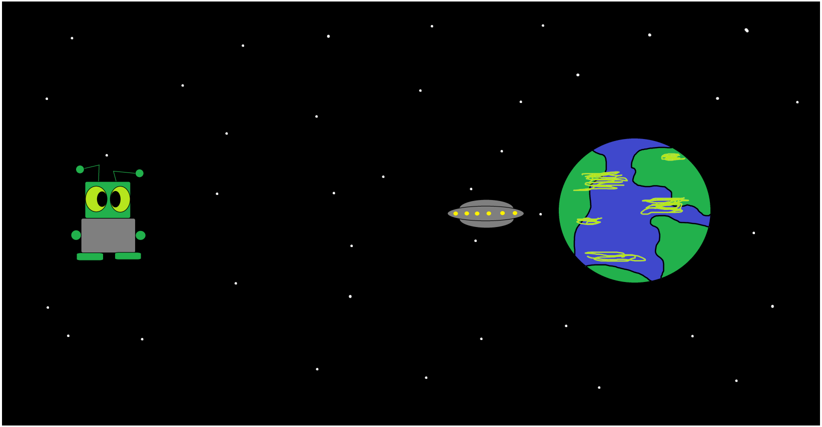

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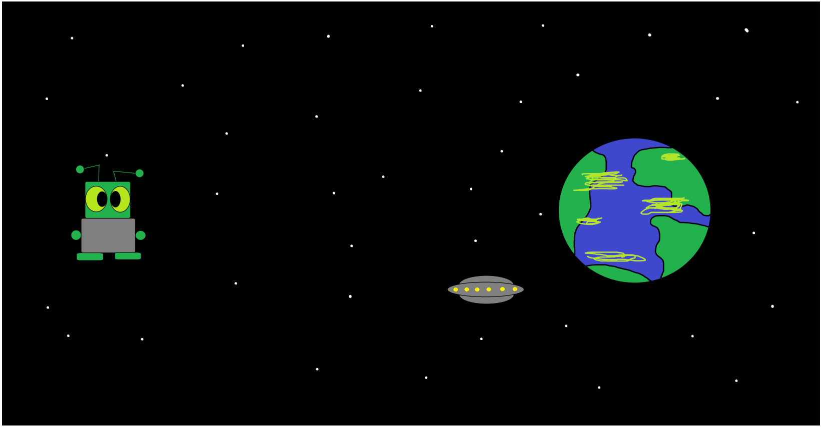

Collatz sequences

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1's occur along the arithmetic progression $9 + 3k$.

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Collatz problem: does the sequence always end in $4, 2, 1$?

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Classical case: **linear recurrences**.

Linear recurrences

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Everyone knows this one ...

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This follows a linear recurrence:

$$a_{n+2} = a_{n+1} + a_n.$$

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The zero set of this recurrence is

$$Z(a_n) = \{2 + 3k : k \geq 0\}$$

an arithmetic progression!

Last example

Set $a_0 = 0$, $a_1 = 1$, and $a_2 = 01$, and define recurisvely

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Zero set: $Z(a_n) = \{0, 2, 3, 7, 16\}$ (finite).

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What about nonlinear recurrences (e.g. polynomial)?

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Every linear recurrence can be viewed as the orbit of $\vec{x_0}$ under a matrix mapping A.

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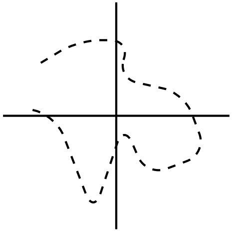

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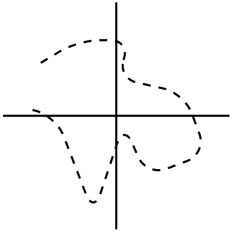

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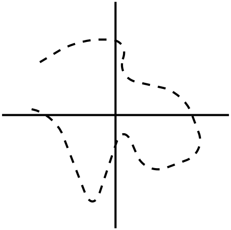

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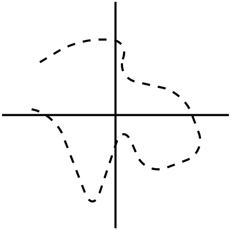

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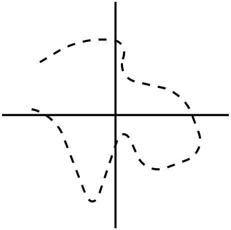

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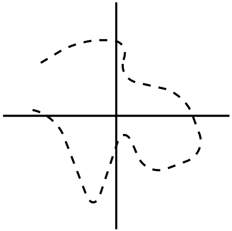

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- What if C is a different closed set (e.g. the unit circle)?

Let X be an algebraic variety, let $C \subseteq X$ be a closed set, let x_0 be a point, and let $\varphi : X \to X$ be a polynomial endomorphism.

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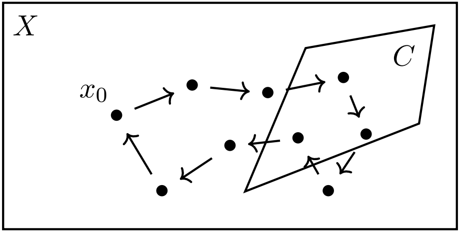

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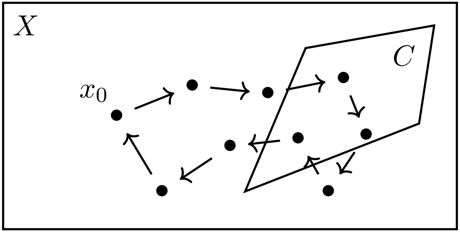

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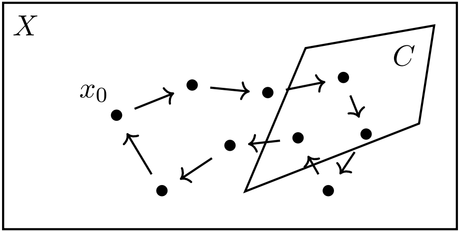

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The Dynamical Mordell–Lang Conjecture:
N is a union of finitely many arithmetic progressions, along with a finite set.

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Theorem (Bell–Ghioca–Tucker, 2014)**.**

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What does this have to do with arithmetic dynamics?

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- Use this infinite progression to break X down into smaller pieces, restrict φ, and proceed by induction.

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Thank you!