The Laplace transform of the lognormal distribution

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Introduction

Definitions

A positive random variable X is said to have a **lognormal distribution**, written $X \sim \text{LN}(\mu, \sigma^2)$, if

$$\ln(X) \sim N(\mu, \sigma^2).$$

The **probability density function** (PDF) of X is

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right], \quad x \in (0, \infty).$$

The lognormal distribution has applications in:

- natural sciences
- finance, actuarial science, and economics
- engineering

Objective

Let $X_i \sim \text{LN}(\mu_i, \sigma_i^2)$, $i = 1, \ldots, n$, be independent and define S_n to be their sum:

$$S_n = X_1 + X_2 + \cdots + X_n. \tag{1}$$

Question: How do we determine the distribution of the r.v. S_n?

Answer: Convolution of CDFs/PDFs.

Problem: These convolutions have no known explicit solution.

Our only option, at present, is to **approximate** the distribution of S_n. Some common methods:

- Moment-matching methods
- Monte Carlo Methods
- **Laplace transform methods**

The Laplace transform method

Suppose a r.v. X has a PDF $f_X(x)$, defined for $x \geq 0$. The **Laplace transform** of X is defined by

$$\varphi_X(z) := \mathbb{E}\left[e^{-zX}\right] = \int_0^\infty e^{-zx} f(x)\, dx \ .$$

for all $z \in \mathbb{C}$ for which the integral is defined. This transform can be used to compute the distribution of $S_n = X_1 + X_2 + \ldots + X_n$ as follows:

1. Compute φ_{X_i}, $i = 1, \ldots, n$.
2. We then have $\varphi_{S_n}(z) = \varphi_{X_1}(z) \cdot \varphi_{X_2}(z) \cdot \ldots \cdot \varphi_{X_n}(z)$.
3. Obtain the PDF via inverse Laplace transform

$$f_{S_n}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{zx} \varphi_{S_n}(z)\ dz, \ \ x > 0.$$

The Laplace transform of the lognormal distribution

Unfortunately, the Laplace transform of a lognormal r.v. X **does not have a known closed form**; again, we must rely on approximations.

We will briefly discuss some results in the literature. Several results focus on the characteristic function defined by

$$\phi_X(t) := \mathbb{E}\left[e^{itX}\right] = \int_0^\infty e^{itx} f(x)\, dx \ , \ \ t \in \mathbb{R}.$$

This is just the restriction of φ_X to the imaginary axis, i.e.

$$\phi_X(t) = \varphi_X(-it).$$

Approaches in the literature

<u>**Series representations**:</u>

- E. Barouch and Gordon M. Kaufman (1976), R. Barakat (1976), P. Holgate (1989), and Leipnik (1991) all investigated the characteristic function ϕ_X.
- Techniques include: power series expansion of lognormal density, the saddle point method, and solving a functional differential equation.
- Some issues include: **Problems with convergence; valid on small domain; success varies with choice of parameters; incorrect results.**

Approaches in the literature

Numerical integration:

- It appears that Gubner (2006): was the first to propose an alternate contour of integration that reduces oscillations of the integrand.
- Tellambura and Senaratne (2010): improved upon Gubner's method by deriving the steepest-descent contour and by providing two, related, closed-form contours.
- **Asmussen et al. (2016) report that the method of Tellambura and Senaratne deliver unreliable results for small σ but works very well for larger values of σ.**

Approaches in the literature

Asmussen et al. (2016):

- Derived a closed-form approximation to the Laplace transform that is asymptotically equivalent as $z \to \infty$.
- They show that

$$\varphi_X(z) = L(z)\left(1 + O(\log^{-1}(z))\right)$$

$$L(z) = \frac{\exp\left[-\frac{W^2(ze^{\mu}\sigma^2)+2W(ze^{\mu}\sigma^2)}{2\sigma^2}\right]}{\sqrt{1+W(ze^{\mu}\sigma^2)}}$$

- **They report the approximation $L(z)$ works well for σ small but accuracy decreases as σ gets larger.**
- They construct a Monte Carlo estimator for the Laplace transform.

Disadvantages with methods in the literature

- The majority of methods are only valid, at most, for arguments in the right half plane, $\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$. As a result, one must exclude some efficient paths of integration when performing an inversion of the Laplace transform.

- It appears that there are no convergent series representations in the literature that are valid on the entire domain of analyticity. Since φ_X is not analytic at the origin, the Taylor series representation centered at any point will have finite radius of convergence.

- Leipnik's result for the characteristic function is incorrect.

Leipnik's result is incorrect

Leipnik's result:

Let $X \sim \mathrm{LN}(0, \sigma^2)$. For $t > 0$ and $0 < k < 1$ the characteristic function is given by

$$\phi_X(t) := \mathbb{E}\left[e^{itX}\right] \stackrel{?}{=} \frac{1}{2\pi} \int_{k-i\infty}^{k+i\infty} \sin(\pi s)\Gamma(s) e^{-(\ln t + i\frac{\pi}{2})s + \frac{\sigma^2}{2}s^2}\; ds.$$

This result is incorrect. To see this:

1. integrand is entire and we can take $k \in \mathbb{R}$.
2. Shift contour to the left of the origin ($k < 0$). Implies $\phi_X(t) = O(t^{|k|})$ as $t \to 0$.
3. Thus, $\phi_X(t) \to 0$ as $t \to 0$.

New approximations and an example

Analytic continuation

Recall that φ_X, for $X \sim \mathrm{LN}(\mu, \sigma^2)$, is given by

$$\varphi_X(z) = \int_0^\infty e^{-zx} f(x)\, dx \; = \int_0^\infty e^{-zx} \left(\frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} \right) dx \; .$$

This is analytic in the right half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > 0\}$.

Proposition

Let $X \sim LN(\mu, \sigma^2)$, and $k > 0$. The Laplace transform of X can be expressed by the integral

$$\varphi_X(z) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s) e^{-(\mu + \ln z)s + \frac{\sigma^2}{2} s^2} \, ds, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \qquad (2)$$

Approximation 1

Proposition

Let $X \sim LN(\mu, \sigma^2)$. Then for any $\alpha \geq 1$,

$$\varphi_X(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} e^{\mu n + \frac{\sigma^2}{2} n^2} \cdot \frac{1}{2} \operatorname{erfc}\left(\frac{\mu + \ln(z/\alpha) + \sigma^2 n}{\sqrt{2}\sigma} \right) + R(z), \quad (3)$$

for $z \in \mathbb{C} \setminus (-\infty, 0]$ and where

$$|R(z)| \leq \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{\pi^2}{2\sigma^2} - \alpha}$$

Furthermore, the series is asymptotic to $\varphi_X(z)$ as $|z| \to 0$.

Proposition

Let $X \sim LN(\mu, \sigma^2)$, *and* $M, N \in \mathbb{N}$. *Then*

$$\varphi_X(z) = \sum_{n=0}^{N} \frac{(-z)^n}{n!} e^{\mu n + \frac{\sigma^2}{2} n^2} \cdot \frac{1}{2} \operatorname{erfc}\left(\frac{\mu + \ln z + \sigma^2 n}{\sqrt{2}\sigma} \right) + \sum_{m=0}^{M} \frac{(-1)^m a_m}{\sqrt{2\pi}\sigma^{m+1}} e^{-\frac{(\mu+\ln z)^2}{2\sigma^2}} H_m\left(-\frac{(\mu + \ln z)}{\sigma} \right) + O(\sigma^{-M-2}),$$

The coefficients a_m *are defined by*

$$a_m = \frac{\Gamma^{(m+1)}(1)}{(m+1)!} + (-1)^{m+1} \cdot \sum_{j=1}^{N} \frac{(-1)^j}{j!} \frac{1}{j^{m+1}},$$

and H_m *is the* m^{th} *(probabilist) Hermite polynomial.*

Application: the density of the r.v. S_n

Recall that the density of the r.v. S_n can be obtained by the inverse Laplace transform:

$$f_{S_n}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \varphi_{S_n}(z) e^{zx} \, dz, \quad x > 0.$$

Proposition

Let $S_n = X_1 + X_2 + \ldots + X_n$, *where* $X_i \sim LN(\mu_i, \sigma_i)$, $i = 1, \ldots, n$ *are independent. Then*

$$f_{S_n}(x) = -\frac{1}{\pi} \int_0^{\infty} \operatorname{Im}\left[\varphi_{S_n}(-t + i \cdot 0)\right] e^{-tx} \, dt, \quad x > 0.$$

where $\varphi_{S_n}(-t + i \cdot 0) = \lim_{\varepsilon \to 0^+} \varphi_{S_n}(-t + i\varepsilon)$,

Example: CDF of the r.v. S_{15}

It can be shown that the CDF of S_n can be obtained via

$$F_{S_n}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\varphi_{S_n}(z)e^{zx}}{z} \; dz, \;\; x > 0.$$

or, using the previous results, one may obtain

$$1 - F_{S_n}(x) = -\frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im}\left[\varphi_{S_n}(-t + i \cdot 0)\right] e^{-tx}}{t} \; dt, \;\; x > 0.$$

Example: We use the above result to compute the CDF of $S_{15} = X_1 + \ldots + X_{15}$, where the underlying r.v.'s are independent and have LN$(0, 0.5)$, LN$(0, 1)$, or LN$(1, 2)$ distribution (say, 5 of each distribution).

Example: CDF of the r.v. S_{15}

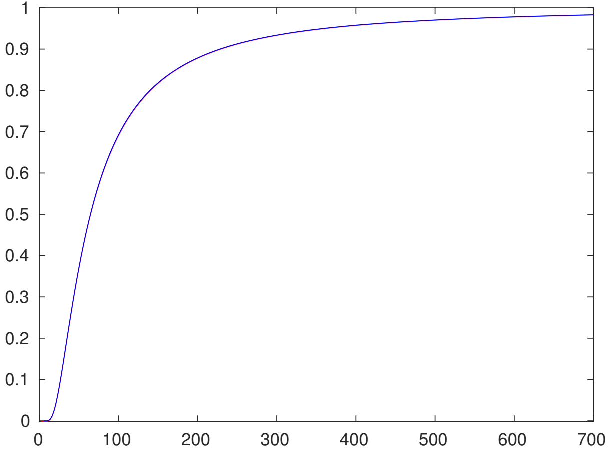

Figure: Approximation of $F_{S_{15}}$ by LTM (red) and Monte Carlo simulation (blue).

Example: CDF of the r.v. S_{15}

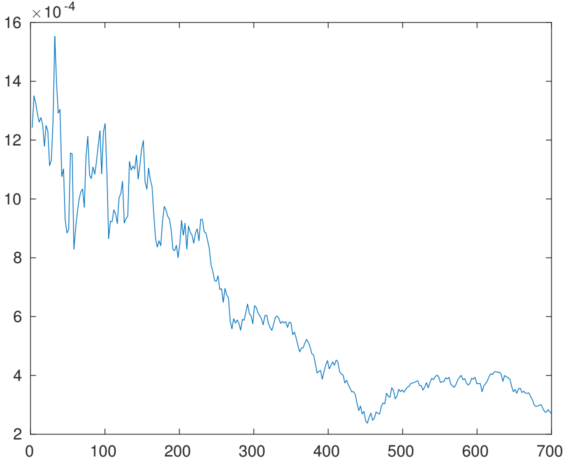

Figure: absolute difference between LTM and Monte Carlo simulation.

Thank you

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