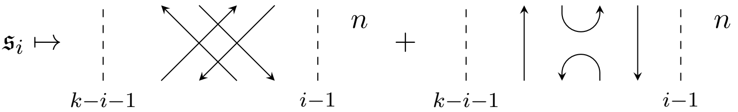
Heisenberg categories and the partition algebra



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Outline

Problem: Relate the partition algebra to Heisenberg categorification by describing an explicit embedding of the partition algebra into certain endomorphism algebras in the Heisenberg category.

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Overview:

1. Presentation of the partition algebra
2. Khovanov's Heisenberg category
3. Planar diagrammatics of the partition algebra
4. Further directions

Presentation of the partition algebra $\mathcal{P}_k(n)$

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Definition 1

Let $k, n \in \mathbb{Z}_{\geq 1}$. The partition algebra $\mathcal{P}_k(n)$ with parameters k and n is the unital associative algebra with basis given by the partition monoid $\mathcal{P}_k$.

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Presentation of $\mathcal{P}_k(n)$ ([HR05, Th. 1.11] and [BH17, Th. 6.5])

The partition algebra $\mathcal{P}_k(n)$ is generated by elements $\mathfrak{s}_i$ $(1 \leq i \leq k-1)$ and $\mathfrak{p}_j$ $(j \in \frac{1}{2}\mathbb{Z}_{\geq 1}, 1 \leq j \leq k)$ of $\mathcal{P}_k$ satisfying the following relations :

Presentation of the partition algebra

$$\mathfrak{s}_i^2 = 1,$$

$$\mathfrak{s}_i\mathfrak{s}_j = \mathfrak{s}_j\mathfrak{s}_i \quad \text{for all } |i-j|>1,$$

$$\mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{s}_i = \mathfrak{s}_{i+1}\mathfrak{s}_i\mathfrak{s}_{i+1},$$

$$\mathfrak{p}_i^2 = \mathfrak{p}_i \quad \text{for all } i \in \frac{1}{2}\mathbb{Z}_{\geq 1},$$

$$\mathfrak{p}_i\mathfrak{p}_j = \mathfrak{p}_j\mathfrak{p}_i \quad \text{for all } |i-j|>\frac{1}{2}, \quad i,j \in \frac{1}{2}\mathbb{Z}_{\geq 1},$$

$$\mathfrak{p}_i\mathfrak{p}_{i\pm\frac{1}{2}}\mathfrak{p}_i = \frac{1}{n}\mathfrak{p}_i \quad \text{for all } i \in \frac{1}{2}\mathbb{Z}_{\geq 1},$$

$$\mathfrak{s}_i\mathfrak{p}_i\mathfrak{p}_{i+1} = \mathfrak{p}_i\mathfrak{p}_{i+1},$$

$$\mathfrak{s}_i\mathfrak{p}_i\mathfrak{s}_i = \mathfrak{p}_{i+1},$$

$$\mathfrak{s}_i\mathfrak{p}_{i+\frac{1}{2}} = \mathfrak{p}_{i+\frac{1}{2}}\mathfrak{s}_i = \mathfrak{p}_{i+\frac{1}{2}},$$

$$\mathfrak{s}_i\mathfrak{s}_{i+1}\mathfrak{p}_{i+\frac{1}{2}}\mathfrak{s}_{i+1}\mathfrak{s}_i = \mathfrak{p}_{i+\frac{3}{2}},$$

$$\mathfrak{s}_i\mathfrak{p}_j = \mathfrak{p}_j\mathfrak{s}_i, \text{ for all } j \in \frac{1}{2}\mathbb{Z}_{\geq 1},\ j \neq i-\frac{1}{2}, i, i+\frac{1}{2}, i+1, i+\frac{3}{2}.$$

Action of $\mathcal{P}_k(n)$

Let $k, n \in \mathbb{Z}_{\geq 1}$. Consider the vector space $V = \mathbb{C}^n$ and the symmetric group S_n.

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Action of S_n and $\mathcal{P}_k(n)$ generators on $V^{\otimes k}$

$$\sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sigma v_1 \otimes \sigma v_2 \otimes \cdots \otimes \sigma v_k, \text{ for all } \sigma \in S_n,$$

$$\mathfrak{p}_i \cdot (v_{\ell_1} \otimes \cdots \otimes v_{\ell_k}) = \frac{1}{n} \sum_{j=1}^{n} v_{\ell_1} \otimes \cdots v_{\ell_{i-1}} \otimes v_j \otimes v_{\ell_{i+1}} \otimes \cdots v_{\ell_k},$$

$$\mathfrak{p}_{i+\frac{1}{2}} \cdot (v_{\ell_1} \otimes \cdots \otimes v_{\ell_k}) = \delta_{\ell_i, \ell_{i+1}} v_{\ell_1} \otimes \cdots \otimes v_{\ell_i} \otimes v_{\ell_{i+1}} \otimes \cdots v_{\ell_k},$$

$$\mathfrak{s}_i \cdot (v_{\ell_1} \otimes \cdots \otimes v_{\ell_k}) = v_{\ell_1} \otimes \cdots v_{\ell_{i-1}} \otimes v_{\ell_{i+1}} \otimes v_{\ell_i} \otimes v_{\ell_{i+2}} \otimes \cdots v_{\ell_k}.$$

Schur-Weyl duality for $\mathcal{P}_k(n)$

It follows that the actions of $\mathcal{P}_k(n)$ and S_n on $V^{\otimes k}$ commute:

$$\mathcal{P}_k(n) \circlearrowright V^{\otimes k} \circlearrowleft S_n$$

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$$\phi_k(n) \colon \mathcal{P}_k(n) \longrightarrow \mathrm{End}_{S_n}(V^{\otimes k}).$$

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Theorem 1 (Schur-Weyl duality)

For $k \in \mathbb{N}$, the map $\phi_k(n)$ is surjective. Moreover, when $n \geq 2k$, $\phi_k(n)$ is also an isomorphism.

Khovanov's Heisenberg category

Definition 2 ([Kho14, §2] and [LS13, §3])

Define the additive $\mathbb{C}$-linear monoidal category as $\mathcal{H}'$ as follows.

- Objects: Direct sum of finite sequence of $+$ and $-$.
- Unit object: $\mathbf{1} = Q_{\emptyset}$.
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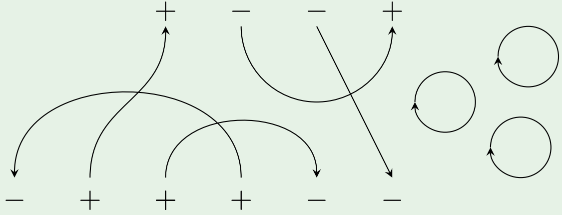
The planar diagrams are oriented compact one-manifolds immersed into the plane strip $\mathbb{R} \times [0,1]$ and where orientation at the endpoints match the signs. The lower and upper endpoints are respectively located at $\{1, \cdots, m\} \times \{0\}$ and $\{1, \cdots, m'\} \times \{1\}$.

Example

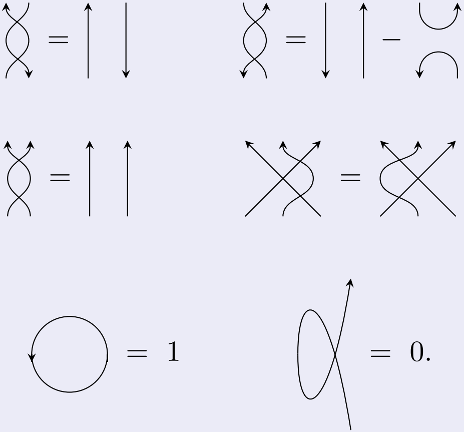
The diagram below is an element of $\mathrm{Hom}_{\mathcal{H}'}(Q_{-+++--}, Q_{+--+})$



Khovanov's Heisenberg category

Local relations

The local relations are:



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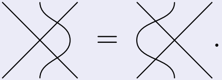
Crossing properties

The local relations of $\mathcal{H}'$ defined above imply the following crossing properties for any orientation of the 3 strands [Kho14, §2.1]:



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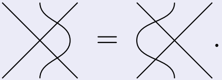

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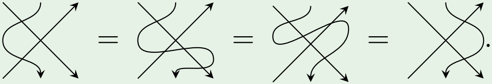

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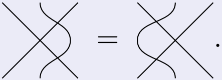

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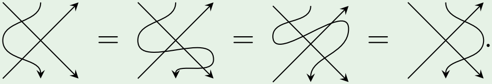
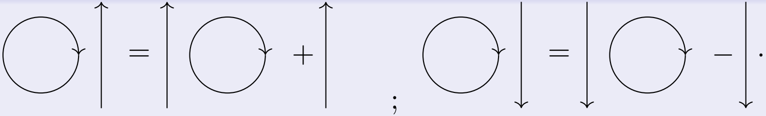


Bubble moves



Endomorphism algebra in $\mathcal{H}'$

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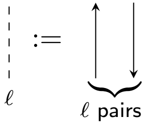
Fix $\epsilon = (+-)^k$.

In sequel all endomorphisms are elements of the algebra endomorphism $\mathrm{End}_{\mathcal{H}'/\mathcal{J}}(Q_\epsilon)$.

Diagrammatic images of the generators of $\mathcal{P}_k(n)$

Notation:



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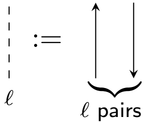
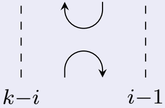
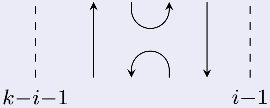
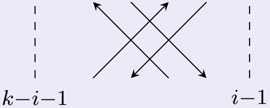
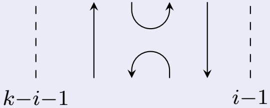


Definition 4

$\mathcal{P}_i = \frac{1}{n}$  ,

$\mathcal{P}_{i+\frac{1}{2}} =$  ,

$\mathcal{S}_i =$ 

$+$  .

Relations of diagrammatic images

Proposition 1 (N. 2018)

$$\mathcal{S}_i^2 = \mathrm{Id}_{Q_\epsilon};$$
$$\mathcal{S}_i\mathcal{S}_j = \mathcal{S}_j\mathcal{S}_i, \quad \text{for } |i-j|>1;$$
$$\mathcal{S}_i\mathcal{S}_{i+1}\mathcal{S}_i = \mathcal{S}_{i+1}\mathcal{S}_i\mathcal{S}_{i+1};$$
$$\mathcal{P}_i^2 = \mathcal{P}_i, \text{ for all } i \in (1/2)\mathbb{Z}_{\geq 1};$$
$$\mathcal{P}_i\mathcal{P}_j = \mathcal{P}_j\mathcal{P}_j, \text{ for all } |i-j|>1/2,\ i,j \in (1/2)\mathbb{Z}_{\geq 1};$$
$$\mathcal{P}_i\mathcal{P}_{i\pm 1/2}\mathcal{P}_i = (1/n)\mathcal{P}_i, \text{ for all } i \in (1/2)\mathbb{Z}_{\geq 1} \text{ and } \mathcal{P}_{1/2} = 1;$$
$$\mathcal{S}_i\mathcal{P}_i\mathcal{P}_{i+1} = \mathcal{P}_i\mathcal{P}_{i+1};$$
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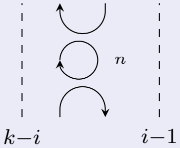
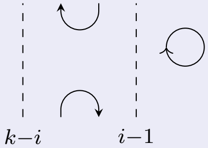
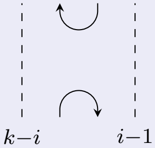
$$\mathcal{S}_i\mathcal{S}_{i+1}\mathcal{P}_{i+1/2}\mathcal{S}_{i+1}\mathcal{S}_i = \mathcal{P}_{i+3/2};$$
$$\mathcal{S}_i\mathcal{P}_j = \mathcal{P}_j\mathcal{S}_i,\ j \in (1/2)\mathbb{Z}_{\geq 1};\ j \neq i-(1/2), i, i+(1/2), i+1, i+(3/2).$$

Proof of relation $\mathcal{P}_i^2 = \mathcal{P}_i$

$\mathcal{P}_i^2 = \frac{1}{n^2}$ 

$= \frac{1}{n^2}$ 

$= \frac{n}{n^2}$ 

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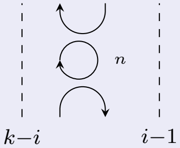

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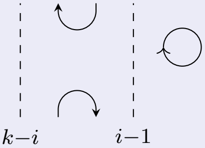

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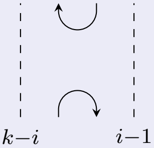

$= \mathcal{P}_i$

Corollary 1 (N. 2018)

There exists an algebra homomorphism $\phi_{k,n}$ from $\mathcal{P}_k(n)$ to the algebra endomorphism $\mathrm{End}_{\mathcal{H}'/\mathcal{J}}(Q_\epsilon)$. More precisely, we have

$$
\begin{aligned}
\phi_{k,n} \colon \mathcal{P}_k(n) &\longrightarrow \mathrm{End}_{\mathcal{H}'/\mathcal{J}}(Q_\epsilon), \\
\mathfrak{s}_i &\longmapsto \mathcal{S}_i, \quad 1 \le i \le n-1, \\
\mathfrak{p}_i &\longmapsto \mathcal{P}_i, \quad i \in \frac{1}{2}\mathbb{Z}_{>0}, \;\; 1 \le i \le n.
\end{aligned}
$$

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Problem 2: Show that the algebra homomorphism $\phi_{k,n}$ is injective.

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Lemma 1

Let $V = \mathbb{C}^n$ be a representation of S_n with the permutation action defined above. For any S_n-module M, we have $\mathrm{Ind}_{n-1}^{n}\mathrm{Res}_{n}^{n-1}(M) \cong M \otimes V$ as S_n-modules.

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By iterating this isomorphism using $S_{n-1} \leq S_n$, we have:

$$(\mathrm{Ind}_{n-1}^{n}\mathrm{Res}_{n}^{n-1})^k(\mathbf{1}_n) \cong V^{\otimes k}.$$

Remark 1

Khovanov in his paper [Kho14] identifies the compositions of induction and restriction functors between categories of modules for symmetric groups with tensor product of bimodules so the natural transformations between them would be the homomorphism of bimodules.

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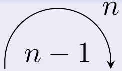
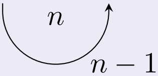
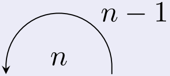
[Kho14, Section. 3.3]



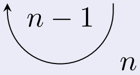
$$: (n)_{n-1}(n) \longrightarrow (n), \quad g \otimes h \mapsto gh$$



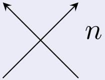
$$: (n-1) \longrightarrow_{n-1} (n)_{n-1}, \quad g \mapsto g$$



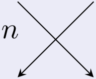
$$: {}_{n-1}(n)_{n-1} \longrightarrow (n-1), \quad g \mapsto \begin{cases} g, & \text{if } g \in S_n \\ 0, & \text{otherwise} \end{cases}$$



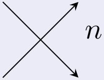
$$(n) \longrightarrow (n)_{n-1}(n), \quad g \mapsto \sum_{i=1}^{n} g g_i \otimes g_i^{-1}$$



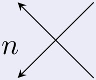
$$: (n+2)_n \longrightarrow (n+2)_n, \quad g \mapsto g s_{n+1}$$



$$: {}_n(n+2) \longrightarrow_n (n+2), \quad g \mapsto s_{n+1} g$$



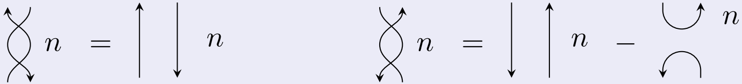
$$: (n)_{n-1}(n) \longrightarrow_n (n+1)_n, \quad g \otimes h \mapsto g s_{n+1} h$$



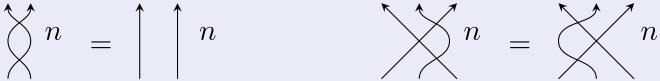
$$: {}_n(n+1)_n \longrightarrow (n)_{n-1}(n), \quad m \mapsto \begin{cases} g \otimes h, & \text{if } m = g s_n h \quad g, h \in \\ 0, & \text{if } m \in S_n \end{cases}$$

Proposition 2 [Kho14, Prop. 7]

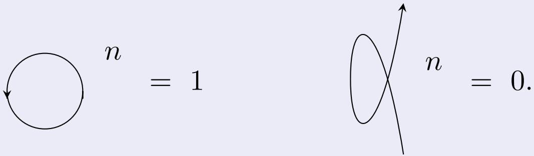
The following relations hold for any $n \in \mathbb{Z}$.



(1)



(2)



(3)

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Example

- $\mathcal{F}'_n(Q_{(+-)^k}) = (\mathrm{Ind}^n_{n-1}\mathrm{Res}^{n-1}_n)^k$.
- The image of $\mathcal{P}_{i+\frac{1}{2}} \in \mathrm{End}_{\mathcal{H}'/\mathcal{J}}(Q_{(+-)^k}) \subset \mathrm{End}_{\mathcal{H}'}(Q_{(+-)^k})$ via $\mathcal{F}'_n$ is:

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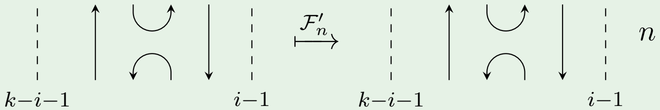

Consider the tensor product functor
$-\otimes_n \mathbf{1}_n : (\Bbbk[S_n], \Bbbk[S_n])$-bimod $\longrightarrow \Bbbk[S_n]$-mod.

$$\begin{aligned} -\otimes_n \mathbf{1}_n \circ \mathcal{F}'_n(Q_\epsilon) &= (n)_{n-1}(n)_{n-1} \cdots (n)_{n-1}(n) \otimes \mathbf{1}_n \\ &\equiv (\mathrm{Ind}_{n-1}^{n}\mathrm{Res}_{n}^{n-1})^k(\mathbf{1}_n) \cong V^{\otimes k}. \end{aligned}$$

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We denote it by $\mathcal{F}_n \colon \mathrm{End}_{\mathcal{H}'/\mathcal{J}}(Q_\epsilon) \longrightarrow \mathrm{End}_{S_n}(V^{\otimes k})$.

Injectivity of the map $\phi_{k,n}$

Theorem 2 (N. 2018)

The following diagram is commutative.

$$\begin{array}{ccc} \mathcal{P}_k(n) & \xrightarrow{\phi_{k,n}} & \mathrm{End}_{\mathcal{H}'/\mathcal{J}}(Q_\epsilon) \\ & \searrow^{\phi_k(n)} & \downarrow \mathcal{F}_n \\ & & \mathrm{End}_{S_n}(V^{\otimes k}) \end{array}$$

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Corollary 2 (N. 2018)

Let $k, n \in \mathbb{Z}_{\geq 0}$ such that $n \geq 2k$. The map $\phi_{k,n}$ is injective.

Further directions

q-Deformation of $\mathcal{H}'(q)$ of the category $\mathcal{H}'$

Define the additive $\Bbbk[q, q^{-1}]$-linear strict monoidal category as $\mathcal{H}'(q)$ as follows.

- Objects: Direct sum of finite sequence of $+$ and $-$.
- Unit object: $\mathbf{1} = Q_{\emptyset}$.
- Morphisms: $\Bbbk[q, q^{-1}]$-module space generated by planar diagrams up to isotopy modulo local relations (different from the one given above).

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Problem: To explore if the arguments using to solve the problems above also work when working with category $\mathcal{H}'(q)$. This corresponds to replacing the group algebra of the symmetric group with the Iwahori-Hecke algebra of type A.

Partition category

Define a monoidal category those objects are all the partitions of $\{1, 2, \ldots, 2k\}$, $k \geq 1$ and construct a (monoidal) functor from this category to the Heisenberg category $\mathcal{H}'$.