

Heisenberg categories and the partition algebra

$$\mathfrak{S}_i \mapsto \begin{array}{c} \text{---} \\ | \\ \text{---} \\ k-i-1 \end{array} \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ i-1 \end{array} n \quad + \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ k-i-1 \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ i-1 \end{array} n$$

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Outline

Problem: Relate the partition algebra to Heisenberg categorification by describing an explicit embedding of the partition algebra into certain endomorphism algebras in the Heisenberg category.

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Overview:

- 1 Presentation of the partition algebra
- 2 Khovanov's Heisenberg category
- 3 Planar diagrammatics of the partition algebra
- 4 Further directions

Presentation of the partition algebra $\mathcal{P}_k(n)$

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Definition 1

Let $k, n \in \mathbb{Z}_{\geq 1}$. The partition algebra $\mathcal{P}_k(n)$ with parameters k and n is the unital associative algebra with basis given by the partition monoid \mathcal{P}_k .

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Presentation of $\mathcal{P}_k(n)$ ([HR05, Th. 1.11] and [BH17, Th. 6.5])

The partition algebra $\mathcal{P}_k(n)$ is generated by elements s_i ($1 \leq i \leq k-1$) and p_j ($j \in \frac{1}{2}\mathbb{Z}_{\geq 1}$, $1 \leq j \leq k$) of \mathcal{P}_k satisfying the following relations :

Presentation of the partition algebra

$$s_i^2 = 1,$$

$$s_i s_j = s_j s_i \quad \text{for all } |i - j| > 1,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

$$p_i^2 = p_i \quad \text{for all } i \in \frac{1}{2}\mathbb{Z}_{\geq 1},$$

$$p_i p_j = p_j p_i \quad \text{for all } |i - j| > \frac{1}{2}, \quad i, j \in \frac{1}{2}\mathbb{Z}_{\geq 1},$$

$$p_i p_{i \pm \frac{1}{2}} p_i = \frac{1}{n} p_i \quad \text{for all } i \in \frac{1}{2}\mathbb{Z}_{\geq 1},$$

$$s_i p_i p_{i+1} = p_i p_{i+1},$$

$$s_i p_i s_i = p_{i+1},$$

$$s_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} s_i = p_{i+\frac{1}{2}},$$

$$s_i s_{i+1} p_{i+\frac{1}{2}} s_{i+1} s_i = p_{i+\frac{3}{2}},$$

$$s_i p_j = p_j s_i, \quad \text{for all } j \in \frac{1}{2}\mathbb{Z}_{\geq 1}, \quad j \neq i - \frac{1}{2}, i, i + \frac{1}{2}, i + 1, i + \frac{3}{2}.$$

Action of $\mathcal{P}_k(n)$

Let $k, n \in \mathbb{Z}_{\geq 1}$. Consider the vector space $V = \mathbb{C}^n$ and the symmetric group S_n .

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Action of S_n and $\mathcal{P}_k(n)$ generators on $V^{\otimes k}$

$$\sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sigma v_1 \otimes \sigma v_2 \otimes \cdots \otimes \sigma v_k, \text{ for all } \sigma \in S_n,$$

$$\mathfrak{p}_i \cdot (v_{\ell_1} \otimes \cdots \otimes v_{\ell_k}) = \frac{1}{n} \sum_{j=1}^n v_{\ell_1} \otimes \cdots \otimes v_{\ell_{i-1}} \otimes v_j \otimes v_{\ell_{i+1}} \otimes \cdots \otimes v_{\ell_k},$$

$$\mathfrak{p}_{i+\frac{1}{2}} \cdot (v_{\ell_1} \otimes \cdots \otimes v_{\ell_k}) = \delta_{\ell_i, \ell_{i+1}} v_{\ell_1} \otimes \cdots \otimes v_{\ell_i} \otimes v_{\ell_{i+1}} \otimes \cdots \otimes v_{\ell_k},$$

$$\mathfrak{s}_i \cdot (v_{\ell_1} \otimes \cdots \otimes v_{\ell_k}) = v_{\ell_1} \otimes \cdots \otimes v_{\ell_{i-1}} \otimes v_{\ell_{i+1}} \otimes v_{\ell_i} \otimes v_{\ell_{i+2}} \otimes \cdots \otimes v_{\ell_k}.$$

Schur-Weyl duality for $\mathcal{P}_k(n)$

It follows that the actions of $\mathcal{P}_k(n)$ and S_n on $V^{\otimes k}$ commute:

$$\mathcal{P}_k(n) \curvearrowright V^{\otimes k} \curvearrowleft S_n$$

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Thus we have defined an algebra homomorphism

$$\phi_k(n): \mathcal{P}_k(n) \longrightarrow \text{End}_{S_n}(V^{\otimes k}).$$

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Theorem 1 (Schur-Weyl duality)

For $k \in \mathbb{N}$, the map $\phi_k(n)$ is surjective. Moreover, when $n \geq 2k$, $\phi_k(n)$ is also an isomorphism.

Khovanov's Heisenberg category

Definition 2 ([Kho14, §2] and [LS13, §3])

Define the additive \mathbb{C} -linear monoidal category as \mathcal{H}' as follows.

- **Objects:** Direct sum of finite sequence of $+$ and $-$.
- **Unit object:** $\mathbf{1} = Q_\emptyset$.
- **Morphisms:** \mathbb{C} -vector space generated by **planar diagrams** up to isotopy and modulo local relations.

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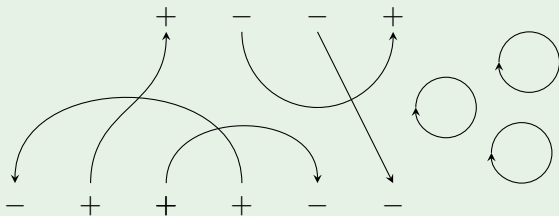
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The planar diagrams are oriented compact one-manifolds immersed into the plane strip $\mathbb{R} \times [0, 1]$ and where orientation at the endpoints match the signs. The lower and upper endpoints are respectively located at $\{1, \dots, m\} \times \{0\}$ and $\{1, \dots, m'\} \times \{1\}$.

Example

The diagram below is an element of $\text{Hom}_{\mathcal{H}'}(Q_{-++++}, Q_{+---+})$



Khovanov's Heisenberg category

Local relations

The local relations are:

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array}$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \downarrow \\ \uparrow \end{array} - \begin{array}{c} \cup \\ \cap \end{array}$$

$$\begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array}$$

$$\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}$$

$$\begin{array}{c} \circlearrowleft \end{array} = 1$$

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \cap \begin{array}{c} \diagdown \\ \diagup \end{array} = 0.$$

Khovanov's Heisenberg category

Crossing properties

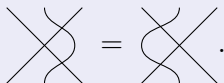
The local relations of \mathcal{H}' defined above imply the following **crossing properties** for any orientation of the 3 strands [Kho14, §2.1]:

The diagram shows an equality between two configurations of three strands. On the left, a crossing occurs between the top and middle strands, with the bottom strand passing through. On the right, a crossing occurs between the middle and bottom strands, with the top strand passing through. The two configurations are separated by an equals sign and a period.

Khovanov's Heisenberg category

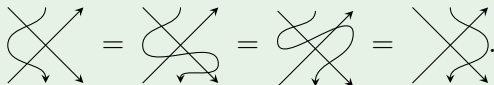
Crossing properties

The local relations of \mathcal{H}' defined above imply the following **crossing properties** for any orientation of the 3 strands [Kho14, §2.1]:



The diagram shows two crossings of three strands. On the left, the strands cross in a specific order. On the right, the strands cross in a different order. The two diagrams are separated by an equals sign, indicating they are equivalent in the Heisenberg category.

Example

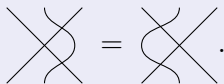


The diagram shows four crossings of three strands, each with arrows indicating orientation. The crossings are connected by equals signs, showing they are all equivalent. The strands are oriented from top-left to bottom-right, top-right to bottom-left, and bottom-left to bottom-right.

Khovanov's Heisenberg category

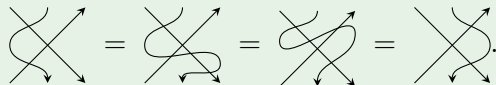
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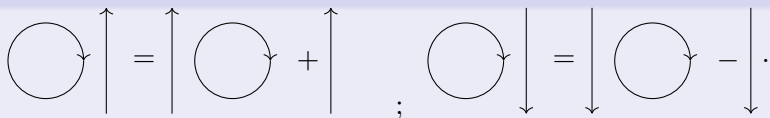
A diagram showing two crossings of three strands. The left side shows a crossing where the top strand crosses over the middle strand, and the middle strand crosses over the bottom strand. The right side shows a crossing where the middle strand crosses over the top strand, and the top strand crosses over the bottom strand. An equals sign is between them.

Example



A sequence of four diagrams showing a crossing of three strands with arrows. The first diagram shows a crossing where the top strand crosses over the middle strand, and the middle strand crosses over the bottom strand. The second diagram shows a crossing where the middle strand crosses over the top strand, and the top strand crosses over the bottom strand. The third diagram shows a crossing where the top strand crosses over the middle strand, and the middle strand crosses over the bottom strand, with a different orientation. The fourth diagram shows a crossing where the middle strand crosses over the top strand, and the top strand crosses over the bottom strand, with a different orientation. All four diagrams are connected by equals signs.

Bubble moves



Two equations involving a circle and a vertical strand. The first equation shows a circle with a clockwise arrow and an upward-pointing strand to its right, equal to an upward-pointing strand to its left and a circle with a clockwise arrow to its right, plus an upward-pointing strand to its right. The second equation shows a circle with a clockwise arrow and a downward-pointing strand to its right, equal to a downward-pointing strand to its left and a circle with a clockwise arrow to its right, minus a downward-pointing strand to its right. A semicolon separates the two equations.

Endomorphism algebra in \mathcal{H}'

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$$D = \bigcirc - n \text{Id}_\emptyset \in \text{End}_{\mathcal{H}'}(Q_\emptyset)$$

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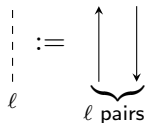
Consider the category \mathcal{H}'/\mathcal{J} .

Fix $\epsilon = (+-)^k$.

In sequel all endomorphisms are elements of the algebra endomorphism $\text{End}_{\mathcal{H}'/\mathcal{J}}(Q_\epsilon)$.

Diagrammatic images of the generators of $\mathcal{P}_k(n)$

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$$\ell \text{ pairs} := \begin{array}{c} \uparrow \\ \downarrow \end{array}$$

Definition 4

$$\mathcal{P}_i = \frac{1}{n} \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array},$$

$k-i \qquad i-1$

$$\mathcal{P}_{i+\frac{1}{2}} = \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array},$$

$k-i-1 \qquad i-1$

$$\mathcal{S}_i = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}.$$

$k-i-1 \qquad i-1 \qquad k-i-1 \qquad i-1$

Relations of diagrammatic images

Proposition 1 (N. 2018)

$$\mathcal{S}_i^2 = \text{Id}_{Q_\epsilon};$$

$$\mathcal{S}_i \mathcal{S}_j = \mathcal{S}_j \mathcal{S}_i, \quad \text{for } |i - j| > 1;$$

$$\mathcal{S}_i \mathcal{S}_{i+1} \mathcal{S}_i = \mathcal{S}_{i+1} \mathcal{S}_i \mathcal{S}_{i+1};$$

$$\mathcal{P}_i^2 = \mathcal{P}_i, \quad \text{for all } i \in (1/2)\mathbb{Z}_{\geq 1};$$

$$\mathcal{P}_i \mathcal{P}_j = \mathcal{P}_j \mathcal{P}_i, \quad \text{for all } |i - j| > 1/2, \quad i, j \in (1/2)\mathbb{Z}_{\geq 1};$$

$$\mathcal{P}_i \mathcal{P}_{i \pm 1/2} \mathcal{P}_i = (1/n) \mathcal{P}_i, \quad \text{for all } i \in (1/2)\mathbb{Z}_{\geq 1} \text{ and } \mathcal{P}_{1/2} = 1;$$

$$\mathcal{S}_i \mathcal{P}_i \mathcal{P}_{i+1} = \mathcal{P}_i \mathcal{P}_{i+1};$$

$$\mathcal{S}_i \mathcal{P}_i \mathcal{S}_i = \mathcal{P}_{i+1};$$

$$\mathcal{S}_i \mathcal{P}_{i+1/2} = \mathcal{P}_{i+1/2} \mathcal{S}_i = \mathcal{P}_{i+1/2};$$

$$\mathcal{S}_i \mathcal{S}_{i+1} \mathcal{P}_{i+1/2} \mathcal{S}_{i+1} \mathcal{S}_i = \mathcal{P}_{i+3/2};$$

$$\mathcal{S}_i \mathcal{P}_j = \mathcal{P}_j \mathcal{S}_i, \quad j \in (1/2)\mathbb{Z}_{\geq 1}; \quad j \neq i - (1/2), i, i + (1/2), i + 1, i + (3/2).$$

Proof of relation $\mathcal{P}_i^2 = \mathcal{P}_i$

$$\mathcal{P}_i^2 = \frac{1}{n^2} \begin{array}{c} \vdots \\ \updownarrow \\ \vdots \end{array} \begin{array}{c} \updownarrow \\ \circlearrowleft \\ \updownarrow \end{array} \begin{array}{c} \vdots \\ \vdots \end{array} = \frac{1}{n^2} \begin{array}{c} \vdots \\ \updownarrow \\ \vdots \end{array} \begin{array}{c} \updownarrow \\ \circlearrowleft \\ \updownarrow \end{array} \begin{array}{c} \vdots \\ \circlearrowleft \\ \vdots \end{array} = \frac{n}{n^2} \begin{array}{c} \vdots \\ \updownarrow \\ \vdots \end{array} \begin{array}{c} \updownarrow \\ \updownarrow \end{array} \begin{array}{c} \vdots \\ \vdots \end{array} = \mathcal{P}_i$$

The diagram shows the proof of the relation $\mathcal{P}_i^2 = \mathcal{P}_i$ using string diagrams. The first term, \mathcal{P}_i^2 , is represented as a fraction $\frac{1}{n^2}$ multiplied by a diagram with two vertical dashed lines labeled $k-i$ and $i-1$. Between these lines, there are two arcs: a top arc pointing right and a bottom arc pointing left. A circle with a counter-clockwise arrow is placed between the two arcs, with a label n to its right. The second term is $\frac{1}{n^2}$ multiplied by a similar diagram, but with a circle with a counter-clockwise arrow placed to the right of the two arcs. The third term is $\frac{n}{n^2}$ multiplied by a diagram where the two arcs are now both pointing right. The final term is \mathcal{P}_i .

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Lemma 1

Let $V = \mathbb{C}^n$ be a representation of S_n with the permutation action defined above. For any S_n -module M , we have $\text{Ind}_{n-1}^n \text{Res}_n^{n-1}(M) \cong M \otimes V$ as S_n -modules.

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By iterating this isomorphism using $S_{n-1} \leq S_n$, we have:

$$(\text{Ind}_{n-1}^n \text{Res}_n^{n-1})^k(\mathbf{1}_n) \cong V^{\otimes k}.$$

Remark 1

Khovanov in his paper [Kho14] identifies the compositions of induction and restriction functors between categories of modules for symmetric groups with tensor product of bimodules so the natural transformations between them would be the homomorphism of bimodules.

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[Kho14, Section. 3.3]

$$\begin{array}{c} \text{---} n \\ \curvearrowright \\ n-1 \end{array} : (n)_{n-1}(n) \longrightarrow (n), \quad g \otimes h \mapsto gh$$

$$\begin{array}{c} n \\ \curvearrowleft \\ \text{---} n-1 \end{array} : (n-1) \longrightarrow_{n-1} (n)_{n-1}, \quad g \mapsto g$$

$$\begin{array}{c} n-1 \\ \curvearrowleft \\ n \end{array} : {}_{n-1}(n)_{n-1} \longrightarrow (n-1), \quad g \mapsto \begin{cases} g, & \text{if } g \in S_n \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{array}{c} n-1 \\ \curvearrowright \\ n \end{array} (n) \longrightarrow (n)_{n-1}(n), \quad g \mapsto \sum_{i=1}^n gg_i \otimes g_i^{-1}$$

$$\begin{array}{c} \nearrow \\ \times \\ \searrow \end{array} n : (n+2)_n \longrightarrow (n+2)_n, \quad g \mapsto gs_{n+1}$$

$$\begin{array}{c} \searrow \\ \times \\ \nearrow \end{array} n : {}_n(n+2) \longrightarrow {}_n(n+2), \quad g \mapsto s_{n+1}g$$

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$$\begin{array}{c} \nearrow \\ \times \\ \searrow \end{array} n : {}_n(n+1)_n \longrightarrow (n)_{n-1}(n), \quad m \mapsto \begin{cases} g \otimes h, & \text{if } m = gs_n h \quad g, h \in S_{n+1} \\ 0, & \text{if } m \in S_n \end{cases}$$

Proposition 2 [Kho14, Prop. 7]

The following relations hold for any $n \in \mathbb{Z}$.

$$\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} n = \begin{array}{c} \uparrow \\ \downarrow \end{array} n \qquad \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} n = \begin{array}{c} \downarrow \\ \uparrow \end{array} n - \begin{array}{c} \cup \\ \cap \end{array} n \quad (1)$$

$$\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} n = \begin{array}{c} \uparrow \\ \uparrow \end{array} n \qquad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} n = \begin{array}{c} \nearrow \\ \searrow \\ \searrow \\ \nearrow \end{array} n \quad (2)$$

$$\begin{array}{c} \circlearrowleft \end{array} n = 1 \qquad \begin{array}{c} \circlearrowright \end{array} n = 0. \quad (3)$$

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Khovanov defines a functor \mathcal{F}'_n from \mathcal{H}' to the category of functors from $S_n\text{-mod}$ to $\bigoplus_m S_m\text{-mod}$,

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Consider the tensor product functor

– $\otimes_n \mathbf{1}_n : (\mathbb{k}[S_n], \mathbb{k}[S_n])\text{-bimod} \longrightarrow \mathbb{k}[S_n]\text{-mod.}$

$$\begin{aligned} - \otimes_n \mathbf{1}_n \circ \mathcal{F}'_n(Q_\epsilon) &= (n)_{n-1} (n)_{n-1} \cdots (n)_{n-1} (n) \otimes \mathbf{1}_n \\ &\cong (\text{Ind}_{n-1}^n \text{Res}_n^{n-1})^k (\mathbf{1}_n) \cong V^{\otimes k}. \end{aligned}$$

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It follows that the functor $- \otimes_n \mathbf{1}_n \circ \mathcal{F}'_n$ defines an algebra homomorphism from $\text{End}_{\mathcal{H}'/\mathcal{J}}(Q_\epsilon)$ to $\text{End}_{S_n}(V^{\otimes k})$.

Consider the tensor product functor

$$- \otimes_n \mathbf{1}_n : (\mathbb{k}[S_n], \mathbb{k}[S_n])\text{-bimod} \longrightarrow \mathbb{k}[S_n]\text{-mod.}$$

$$\begin{aligned} - \otimes_n \mathbf{1}_n \circ \mathcal{F}'_n(Q_\epsilon) &= (n)_{n-1}(n)_{n-1} \cdots (n)_{n-1}(n) \otimes \mathbf{1}_n \\ &\cong (\text{Ind}_{n-1}^n \text{Res}_n^{n-1})^k(\mathbf{1}_n) \cong V^{\otimes k}. \end{aligned}$$

It follows that the functor $- \otimes_n \mathbf{1}_n \circ \mathcal{F}'_n$ defines an algebra homomorphism from $\text{End}_{\mathcal{H}'/\mathcal{J}}(Q_\epsilon)$ to $\text{End}_{S_n}(V^{\otimes k})$.

We denote it by $\mathcal{F}_n: \text{End}_{\mathcal{H}'/\mathcal{J}}(Q_\epsilon) \longrightarrow \text{End}_{S_n}(V^{\otimes k})$.

Injectivity of the map $\phi_{k,n}$

Theorem 2 (N. 2018)

The following diagram is commutative.

$$\begin{array}{ccc} \mathcal{P}_k(n) & \xrightarrow{\phi_{k,n}} & \text{End}_{\mathcal{H}'/\mathcal{J}}(Q_\epsilon) \\ & \searrow \phi_k(n) & \downarrow \mathcal{F}_n \\ & & \text{End}_{S_n}(V^{\otimes k}) \end{array}$$

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Corollary 2 (N. 2018)

Let $k, n \in \mathbb{Z}_{\geq 0}$ such that $n \geq 2k$. The map $\phi_{k,n}$ is injective.

Further directions

q -Deformation of $\mathcal{H}'(q)$ of the category \mathcal{H}'

Define the additive $\mathbb{k}[q, q^{-1}]$ -linear strict monoidal category as $\mathcal{H}'(q)$ as follows.

- **Objects:** Direct sum of finite sequence of $+$ and $-$.
- **Unit object:** $\mathbf{1} = Q_\emptyset$.
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Problem: To explore if the arguments using to solve the problems above also work when working with category $\mathcal{H}'(q)$. This corresponds to replacing the group algebra of the symmetric group with the Iwahori-Hecke algebra of type A .

Partition category

Define a monoidal category whose objects are all the partitions of $\{1, 2, \dots, 2k\}$, $k \geq 1$ and construct a (monoidal) functor from this category to the Heisenberg category \mathcal{H}' .