Entanglement breaking rank

Jitendra Prakash
Department of Pure Mathematics and Institute for Quantum Computing
University of Waterloo

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(With Vern I. Paulsen, Mizanur Rahaman, Satish Pandey)

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Outline

1. Two conjectures

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6. Other directions

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Conjecture (Zauner)

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Definition (Mutually unbiased bases)

Two orthonormal bases $\{e_1, ..., e_d\}$ and $\{f_1, ..., f_d\}$ of $\mathbb{C}^d$ are called *mutually unbiased* if

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Conjecture (MUB)

For each $d \geq 2$, *there exist* $d+1$ *mutually unbiases bases in* $\mathbb{C}^d$.

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- Both have important applications in quantum information theory, for example, in quantum cryptography and quantum tomography.

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is positive semi-definite.

(b) *trace preserving*: $\mathrm{Tr}(\Phi(X)) = \mathrm{Tr}(X)$, for all $X \in \mathbb{M}_d$.

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Definition (Choi rank)

The Choi rank of a quantum channel $\Phi : \mathbb{M}_d \to \mathbb{M}_d$ is the minimum number of operators $B_1, \ldots, B_K$ required in its Choi-Kraus representation, and is denoted by $\mathrm{cr}(\Phi)$.

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Entanglement breaking maps

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Definition (Horodecki, Shor, Ruskai (2003))

A quantum channel $\Phi : \mathbb{M}_d \to \mathbb{M}_d$ is called *entanglement breaking* if its Choi matrix

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where each R_k is a rank one matrix.

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Clearly, $\mathrm{ebr}(\Phi) \geq \mathrm{cr}(\Phi)$, where $\mathrm{cr}(\Phi)$ is the Choi rank of Φ.

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In quantum information such a set of d^2 rank one projections $\{P_i\}_{i=1}^{d^2}$ is called a *symmetric informationally complete positive operator-valued measure* (SIC POVM).

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Can be shown independently: Z is entanglement breaking.

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Channel from SIC POVM

If there exists a SIC POVM $\{P_i\}_{i=1}^{d^2} \subset \mathbb{M}_d$, consider the map $\Phi : \mathbb{M}_d \to \mathbb{M}_d$,

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MUB conjecture (again)

Suppose there are $d+1$ orthonormal bases of $\mathbb{C}^2$,

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Question: Converse?

Other convex combinations

Theorem

Let $t \in \left[0, \frac{1}{d+1}\right]$*. If* $\Phi_t = t\mathcal{I}_d + (1-t)\Psi_d$*, then* Φ_t *is entanglement breaking.*

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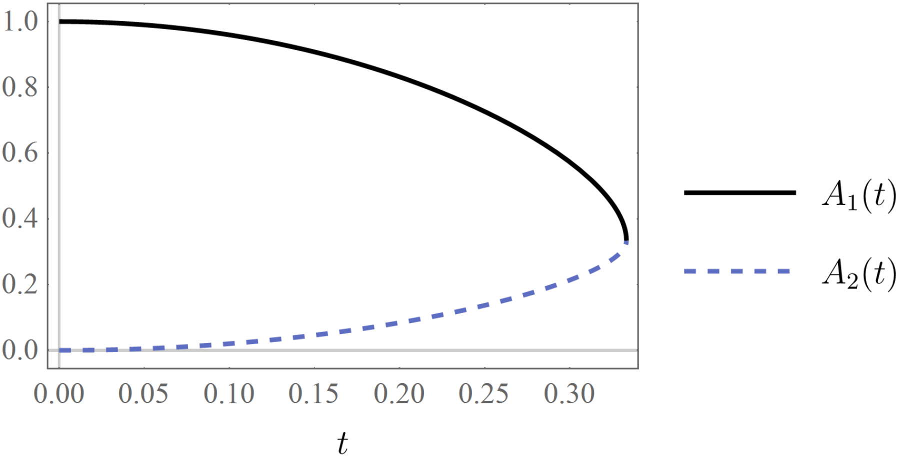
Conjecture

There exist a continuous family of d^2 *rank-one matrices* $R_i(t)$ *such that*

$$\Phi_t(X) = \sum_{i=1}^{d^2} R_i(t) X R_i(t)^*,$$

for all $0 \leq t \leq \frac{1}{d+1}$.

$d = 2$



Thank you!