Hybridizable Discontinuous Galerkin Methods for Linear Free Surface Problems

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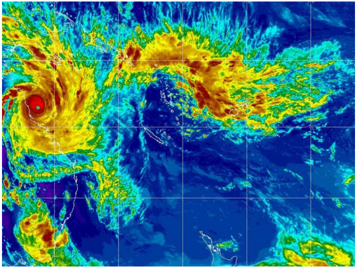
Design of ships

Design of levees and sea walls

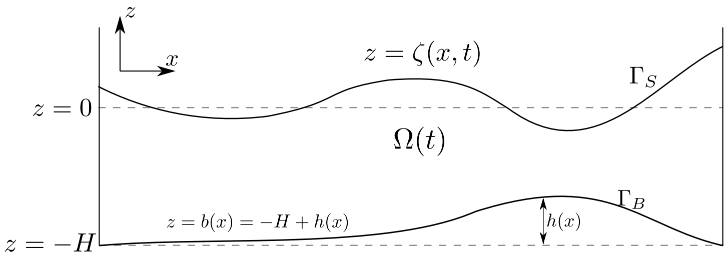
Design of offshore structures



Storm tracking

Notation

- $\zeta(x, y, t)$ represents the wave height
- $\Gamma_S(t) := \{(t, x, y, z) \in \mathbb{R}^4 | f_S(x, y, z, t) = 0\}$ is the free surface, where $f_S(x, y, z, t) = \zeta(x, y, t) - z$
- $\Gamma_B := \{(x, y, z) \in \mathbb{R}^3 | f_B(x, y, z) = 0\}$ is the sea bottom, where $f_B(x, y, z) = b(x, y) - z$
- $\mathbf{u} = (u, v, w)$ is the velocity field of the fluid.



Free surface boundary conditions

Kinematic condition

The free surface moves with the fluid

$$\frac{D}{Dt}\left[\zeta(x,y,t)-z\right]=0,$$

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Kinematic condition

The free surface moves with the fluid

$$\frac{D}{Dt}\left[\zeta(x,y,t) - z\right] = 0,$$

$$\boxed{\partial_t f_S + \nabla f_S \cdot \mathbf{u} = 0 \quad \text{on } z = \zeta(x,y,t)}$$

Free surface boundary conditions

Dynamic condition

Equilibrium of forces

$$\boxed{P = P_a \quad \text{on } z = \zeta(x, y, t)}$$

Free surface boundary conditions

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For irrotational flow, there exists ϕ such that $\mathbf{u} = \nabla\phi$, and

$$\boxed{\partial_t \phi + \frac{1}{2}\nabla\phi \cdot \nabla\phi + g\zeta = 0 \quad \text{on } z = \zeta(x, y, t)}$$

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If the wave amplitude is small, these conditions can be **linearized**.

Linearized free surface boundary conditions

For small amplitude waves, in the case of irrotational flow,

Linearized kinematic condition

$$\nabla\phi \cdot \mathbf{n} = \partial_t \zeta \quad \text{on } z = 0$$

Linearized dynamic condition

$$\partial_t \phi + g\zeta = 0 \quad \text{on } z = 0$$

These conditions are applied at a **fixed boundary**, $z = 0$. The domain is fixed in time.

A particular water wave model

A particular water wave model

Irrotational flow, linearized BC's

$$\begin{cases} -\Delta\phi = 0 & \text{in } \Omega \\ \nabla\phi \cdot \mathbf{n} = \partial_t\zeta & \text{on } z = 0 \\ \partial_t\phi + \zeta = 0 & \text{on } z = 0 \\ \nabla\phi \cdot \mathbf{n} = 0 & \text{on } \Gamma_B \end{cases}$$

plus some extra periodic boundary conditions on the vertical edges of the domain. Here, $\mathbf{u} = \nabla\phi$.

HDG

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- A variable that lives only on the facets is introduced
- The only communication between two neighbouring elements is through this facet variable
- Linear system only for facet variables: smaller than for DG

HDG for second-order elliptic problems [B. Cockburn, B. Dong and J. Guzman, 2008]

- For ϕ_h and $\mathbf{q}_h$ of degree p: $\|\phi_h - \phi\|_{L^2(\Omega)} \sim O(h^{p+1})$, $\|\mathbf{q}_h - \nabla\phi\|_{L^2(\Omega)} \sim O(h^{p+1})$
- Element-wise postprocessing: $\|\phi_h^* - \phi\|_{L^2(\Omega)} \sim O(h^{p+2})$

Introduce a variable $\mathbf{q} = -\nabla\phi$ and combine the two free surface BC's:

Rewritten equations

$$\begin{aligned} \mathbf{q} + \nabla\phi &= 0 \ \text{ in } \Omega, \\ \nabla \cdot \mathbf{q} &= 0 \ \text{ in } \Omega, \\ \partial_t^2\phi - \mathbf{q}\cdot\mathbf{n} &= 0 \ \text{ on } \Gamma_S, \\ \mathbf{q}\cdot\mathbf{n} &= 0 \ \text{ on } \Gamma_B, \end{aligned}$$

with periodic BC's on the rest of the boundaries.

Apply the HDG method[1] in space, and a second-order BDF scheme in time:

Introduce the finite element spaces

$$W_h^p := \{w \in L^2(\Omega) \,:\, w|_K \in W^p(K), \forall K \in \mathcal{T}_h\},$$
$$\mathbf{V}_h^p := \{\mathbf{v} \in \left[L^2(\Omega)\right]^d \,:\, \mathbf{v}|_K \in \mathbf{V}^p(K), \forall K \in \mathcal{T}_h\}.$$

1 [B. Cockburn, B. Dong and J. Guzman, 2008]

Multiply $\mathbf{q} + \nabla\phi = 0$ by $\mathbf{v_h} \in \mathbf{V}_h^p$, and $\nabla \cdot \mathbf{q} = 0$ by $w_h \in W_h^p$, integrate over $K \in \mathcal{T}_h$, and apply integration by parts:

$$\begin{aligned} \int_K \mathbf{q} \cdot \mathbf{v_h} dx - \int_K \phi \nabla \cdot \mathbf{v_h} dx + \int_{\partial K} \widehat{\phi}\, \mathbf{v}_h \cdot \mathbf{n} ds &= 0, \\ - \int_K \nabla w_h \cdot \mathbf{q} dx + \int_{\partial K} w_h \widehat{\mathbf{q}} \cdot \mathbf{n} ds &= 0. \end{aligned}$$

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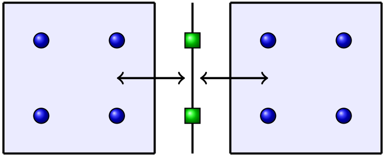

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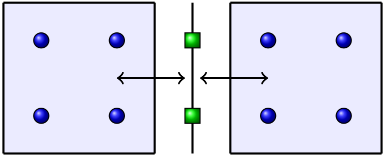

Trace finite element space

$$M_h^p := \{\mu \in L^2(\mathcal{E}_h) \, : \, \mu|_e \in \mathcal{P}^p(e), \, \forall e \in \mathcal{E}_h\}$$

Continuity of the numerical flux $\widehat{\mathbf{q}}$ in the normal direction across faces:

$$\langle [\![\widehat{\mathbf{q}} \cdot \mathbf{n}]\!], \mu_h \rangle_{\mathcal{E}_h} = \langle \partial_t^2 \phi, \mu_h \rangle_{\Gamma_S}, \qquad \forall \mu_h \in M_h^p.$$

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For all $(w_h,\, \mathbf{v}_h,\, \mu_h) \in W_h^p \times \mathbf{V}_h^p \times M_h^p$:

$$\begin{aligned}
-\left(\mathbf{q}, \mathbf{v}_h\right)_{\mathcal{T}_h} + \left(\phi, \nabla \cdot \mathbf{v}_h\right)_{\mathcal{T}_h} - \langle \lambda, \mathbf{v}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \mathbf{v}_h \in \mathbf{V}_h^p \\
\left(w_h, \nabla \cdot \mathbf{q}\right)_{\mathcal{T}_h} + \tau \langle w_h, \phi \rangle_{\partial \mathcal{T}_h} - \tau \langle w_h, \lambda \rangle_{\partial \mathcal{T}_h} &= 0, && \forall w_h \in W_h^p, \\
-\langle \mathbf{q} \cdot \mathbf{n}, \mu_h \rangle_{\partial \mathcal{T}_h} - \langle \tau \phi, \mu_h \rangle_{\partial \mathcal{T}_h} + \langle \tau \lambda, \mu_h \rangle_{\partial \mathcal{T}_h} &= \langle \partial_t^2 \phi, \mu_h \rangle_{\Gamma_S}, && \forall \mu_h \in M_h^p.
\end{aligned}$$

A first order BDF for $n = 1$, and a second order BDF for $n \geq 2$:

For $n = 1$

$$\partial_t^2 \phi\Big|_{t_1} \approx \frac{1}{\Delta t^2}\phi^1 - \frac{1}{\Delta t^2}\phi^0 - \frac{1}{\Delta t}\sigma^0,$$

for $n \geq 2$

$$\partial_t^2 \phi\Big|_{t_n} \approx \frac{9}{4\Delta t^2}\phi^n - \frac{3}{\Delta t^2}\phi^{n-1} + \frac{3}{4\Delta t^2}\phi^{n-2} - \frac{2}{\Delta t}\sigma^{n-1} + \frac{1}{2\Delta t}\sigma^{n-2}.$$

Fully discrete weak form

Find
$(\phi_h^n, \mathbf{q}_h^n, \lambda_h^n) \in W_h^p \times \mathbf{V}_h^p \times M_h^p$ s.t. for all $(w_h, \mathbf{v}_h, \mu_h) \in W_h^p \times \mathbf{V}_h^p \times M_h^p$, the following relations are satisfied for $n = 1$

$$
\begin{aligned}
-\left(\mathbf{q}_h^1, \mathbf{v}_h\right)_{\mathcal{T}_h} + \left(\phi_h^1, \nabla \cdot \mathbf{v}_h\right)_{\mathcal{T}_h} - \langle \lambda_h^1, \mathbf{v}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} &= 0, \\
\left(w_h, \nabla \cdot \mathbf{q}_h^1\right)_{\mathcal{T}_h} + \tau \langle w_h, \phi_h^1 \rangle_{\partial \mathcal{T}_h} - \tau \langle w_h, \lambda_h^1 \rangle_{\partial \mathcal{T}_h} &= 0, \\
-\langle \mathbf{q}_h^1 \cdot \mathbf{n}, \mu_h \rangle_{\mathcal{E}_h} - \langle \tau \phi_h^1, \mu_h \rangle_{\mathcal{E}_h} + \langle \tau \lambda_h^1, \mu_h \rangle_{\mathcal{E}_h} + \frac{1}{\Delta t^2} \langle \phi_h^1, \mu_h \rangle_{\Gamma_S} &= \\
\frac{1}{\Delta t^2} \langle \phi_h^0, \mu_h \rangle_{\Gamma_S} + \frac{1}{\Delta t} \langle \sigma_h^0, \mu_h \rangle_{\Gamma_S}&,
\end{aligned}
$$

[1] [B. Cockburn, B. Dong and J. Guzman, 2008]

and for $n \geq 2$:

$$
\begin{aligned}
-(\mathbf{q}_h^n, \mathbf{v}_h)_{\mathcal{T}_h} + (\phi_h^n, \nabla \cdot \mathbf{v}_h)_{\mathcal{T}_h} - \langle \lambda_h^n, \mathbf{v}_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0, \\
(w_h, \nabla \cdot \mathbf{q}_h^n)_{\mathcal{T}_h} + \tau \langle w_h, \phi_h^n \rangle_{\partial \mathcal{T}_h} - \tau \langle w_h, \lambda_h^n \rangle_{\partial \mathcal{T}_h} = 0, \\
-\langle \mathbf{q}_h^n \cdot \mathbf{n}, \mu_h \rangle_{\mathcal{E}_h} - \langle \tau \phi_h^n, \mu_h \rangle_{\mathcal{E}_h} + \langle \tau \lambda_h^n, \mu_h \rangle_{\mathcal{E}_h} + \frac{9}{4\Delta t^2} \langle \phi_h^n, \mu_h \rangle_{\Gamma_S} = \\
\frac{3}{\Delta t^2} \langle \phi_h^{n-1}, \mu_h \rangle_{\Gamma_S} - \frac{3}{4\Delta t^2} \langle \phi_h^{n-2}, \mu_h \rangle_{\Gamma_S} + \\
\frac{2}{\Delta t} \langle \sigma_h^{n-1}, \mu_h \rangle_{\Gamma_S} - \frac{1}{2\Delta t} \langle \sigma_h^{n-2}, \mu_h \rangle_{\Gamma_S}.
\end{aligned}
$$

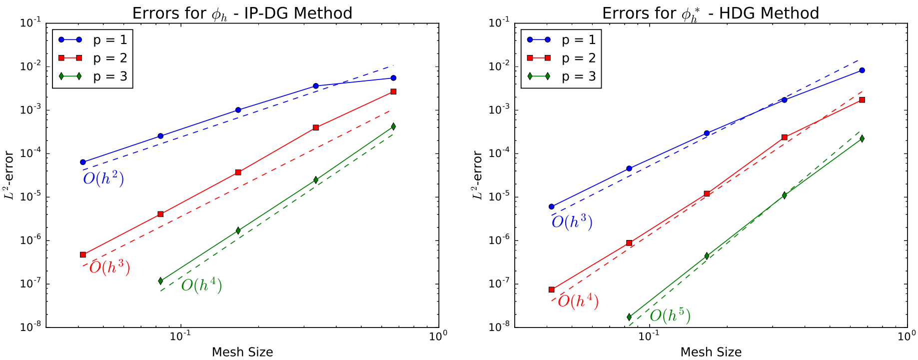
Take $\Omega = [-1, 1] \times [-1, 0]$. The analytical solution is

$$\phi(x, y, t) = \phi_0 \cosh(k(y+1)) \cos(\omega t - kx),$$

where ϕ_0 is the amplitude of the velocity potential, k is the wave number and ω is the frequency of the oscillations.

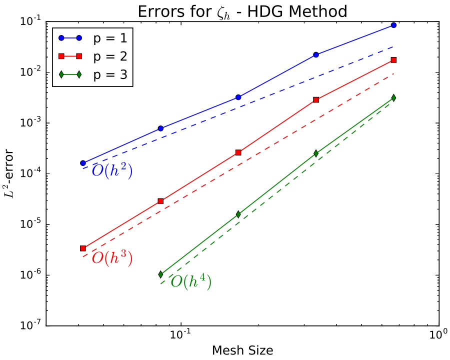
The wave height $\zeta(x, t)$ is

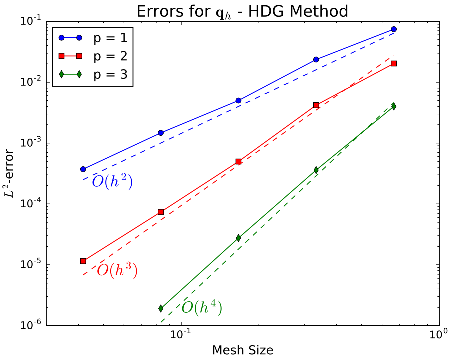
$$\zeta(x, t) = -\partial_t \phi(x, 0, t) = \phi_0 \omega \cosh(k) \sin(\omega t - kx).$$



Error in the L^2 norm is $O(h^{p+1} + \Delta t^2)$. For $p > 1$, Δt has to be **very small**.

[J. J. W. van der Vegt and S. K. Tomar, 2005]


Errors for ζ_h - HDG Method
p = 1
p = 2
p = 3
L^2-error
Mesh Size
$O(h^2)$
$O(h^3)$
$O(h^4)$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{0}


Errors for $\mathbf{q}_h$ - HDG Method
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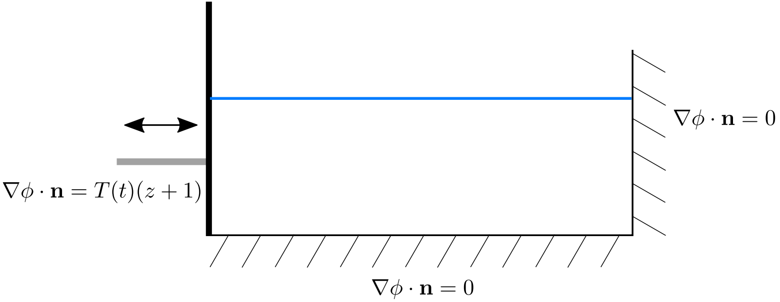
Take $\Omega = [0, 10] \times [-1, 0]$.



Conclusions

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- Only second order in time

Future work

- Consider more complicated cases where the domain is time-dependent $\Rightarrow$ space-time methods

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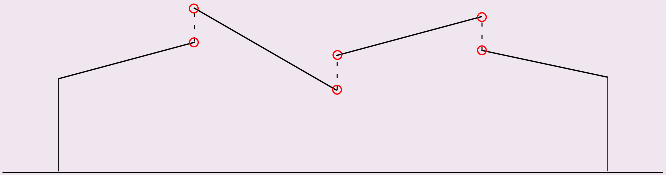

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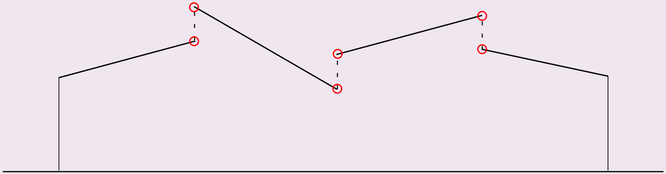


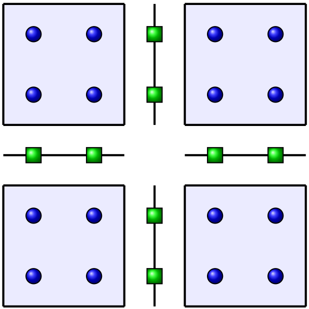
- Apply space-time Embedded DG (EDG) to discretize the free surface boundary conditions

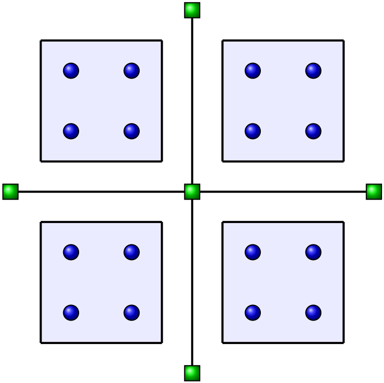
Hybridizable DG (HDG)

$$M_h^p := \{\mu \in L^2(\mathcal{E}_h) \; : \; \mu|_e \in \mathcal{P}^p(e), \; \forall e \in \mathcal{E}_h\}.$$

Embedded DG (EDG)

$$M_h^{p^*} := M_h^p \cap C^0(\mathcal{E}_h)$$





Thank you!

Postprocessing

Find $\phi_h^* \in V_h^{p+1}$ such that it minimizes $|\nabla \phi_h^* + \mathbf{q}_h|$

$$(1, \phi_h^*)_K = (1, \phi_h)_K \,,$$
$$(\nabla w_h^*, \nabla \phi_h^*)_K = -\,(\nabla w_h^*, \mathbf{q}_h)_K \qquad \forall w_h^* \in W_h^{p+1}.$$

Linear System

$$\begin{bmatrix} A & B^T & C^T \\ B & D & E \\ C & G & H \end{bmatrix} \begin{bmatrix} Q \\ U \\ \Lambda \end{bmatrix} = \begin{bmatrix} R \\ F \\ L \end{bmatrix}$$

A, B, D are block diagonal, then

$$\begin{bmatrix} Q \\ U \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & D \end{bmatrix}^{-1} \left(\begin{bmatrix} R \\ F \end{bmatrix} - \begin{bmatrix} C^T \\ E \end{bmatrix} \Lambda \right)$$

and

$$CQ + GU + H\Lambda = L$$